

# Notes on the Bloch Sphere

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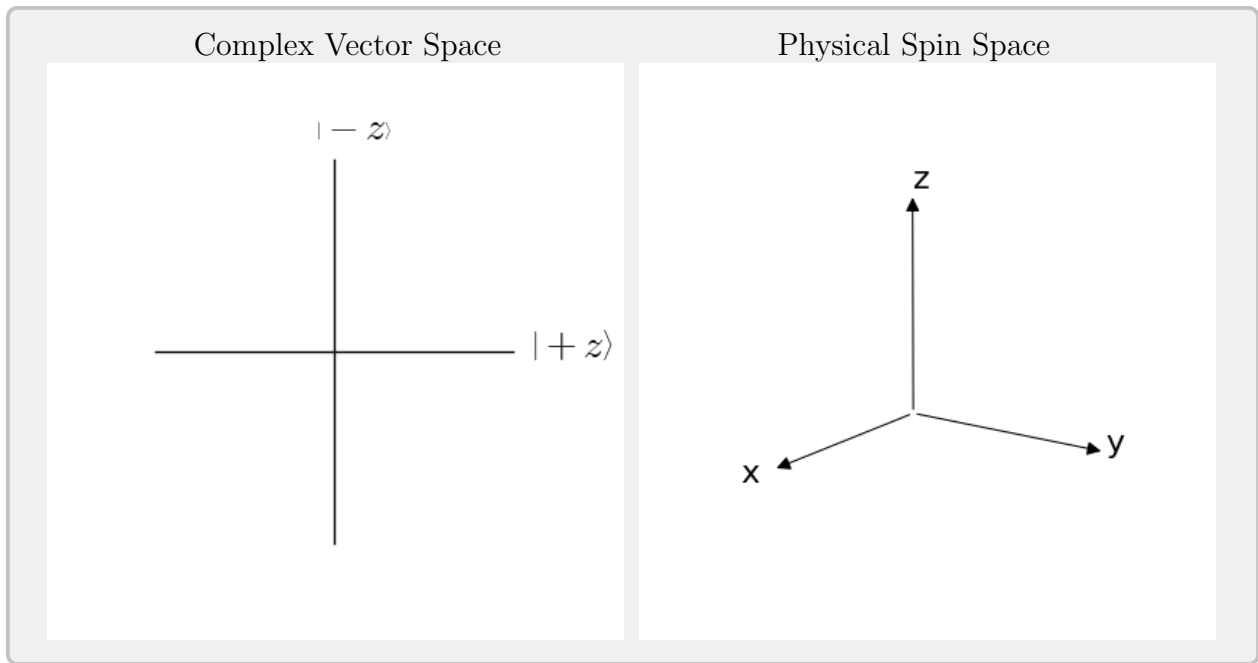
# Contents

<b>1</b>	<b>Introducing the Bloch Sphere or Spin Vector space</b>	<b>1</b>
1.1	Review of the 3D spin vector . . . . .	1
1.2	The Bloch Sphere is just the spin vector space . . . . .	2
<b>2</b>	<b>Translating between the two spaces</b>	<b>3</b>
2.1	Going from Spin Space to Vector Space . . . . .	4
2.2	Going from Vector Space to Spin Space . . . . .	6
2.3	Creating the observable matrix from a state vector . . . . .	7
2.4	Creating the observable matrix from a density matrix . . . . .	8
<b>3</b>	<b>Some Examples</b>	<b>9</b>
<b>4</b>	<b>Summary of Formulae</b>	<b>11</b>
4.1	The Completeness Relation . . . . .	11
4.2	Constructing an observable matrix (general case) . . . . .	11
4.3	Constructing an 1-bit “observable” matrix (including mixed states) . . . . .	11
4.4	Decomposing the observable matrix . . . . .	11
4.5	To get the angles of the spin vector from the state . . . . .	11
<b>5</b>	<b>Finding Eigenvalues and Eigenvectors</b>	<b>12</b>
5.1	Finding eigenvectors on the computer . . . . .	12
5.2	Finding eigenvectors by hand . . . . .	13
<b>6</b>	<b>Unitary operations as rotations on the Bloch sphere</b>	<b>13</b>

# 1 Introducing the Bloch Sphere or Spin Vector space

## 1.1 Review of the 3D spin vector

Last year (in section 6.2 of the “old notes”) we worked with an arbitrary spin vector in physical 3-D space, and learned how to calculate the equivalent quantum state in our mathematical model which is a 2-D complex vector space.



Let's just review that now. In the 3-D physical space we have a spin vector  $\vec{S} = \begin{pmatrix} S_x \\ S_y \\ S_z \end{pmatrix}$ .

The three components of  $\vec{S}$  are simply the projections of that vector along the  $X, Y$ , and  $Z$  axes. There are some obvious relationships between spin vectors and quantum states. For example:

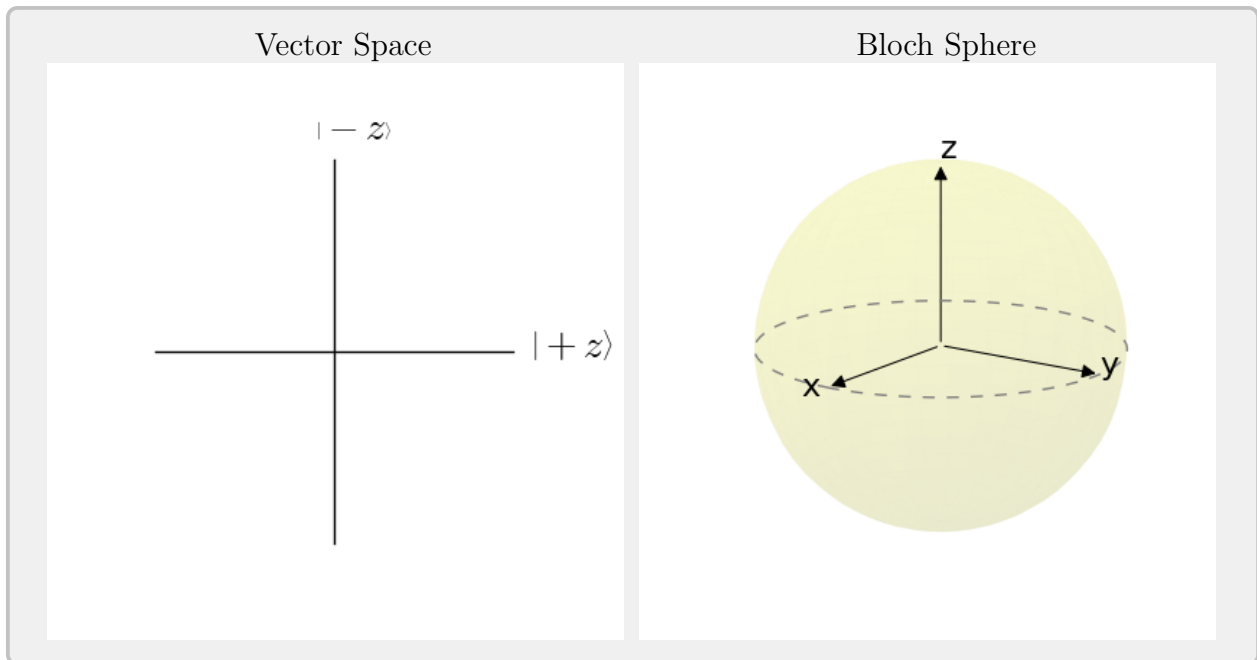
$$| + x \rangle \mapsto \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad | + y \rangle \mapsto \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad | + z \rangle \mapsto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and so on.}$$

In the complex vector space an arbitrary (1 bit) state is a column vector with two complex components:  $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ . Of course, we can't really plot the complex space, so we typically just

draw the subset of this space that corresponds to a real plane. That's what I've done above. This plane is sufficient to plot any states that represent spin vectors in the  $XZ$  plane in the 3-D spin space.

## 1.2 The Bloch Sphere is just the spin vector space

Now the Bloch Sphere is *simply a unit circle drawn around the origin in the 3-D spin vector space*.

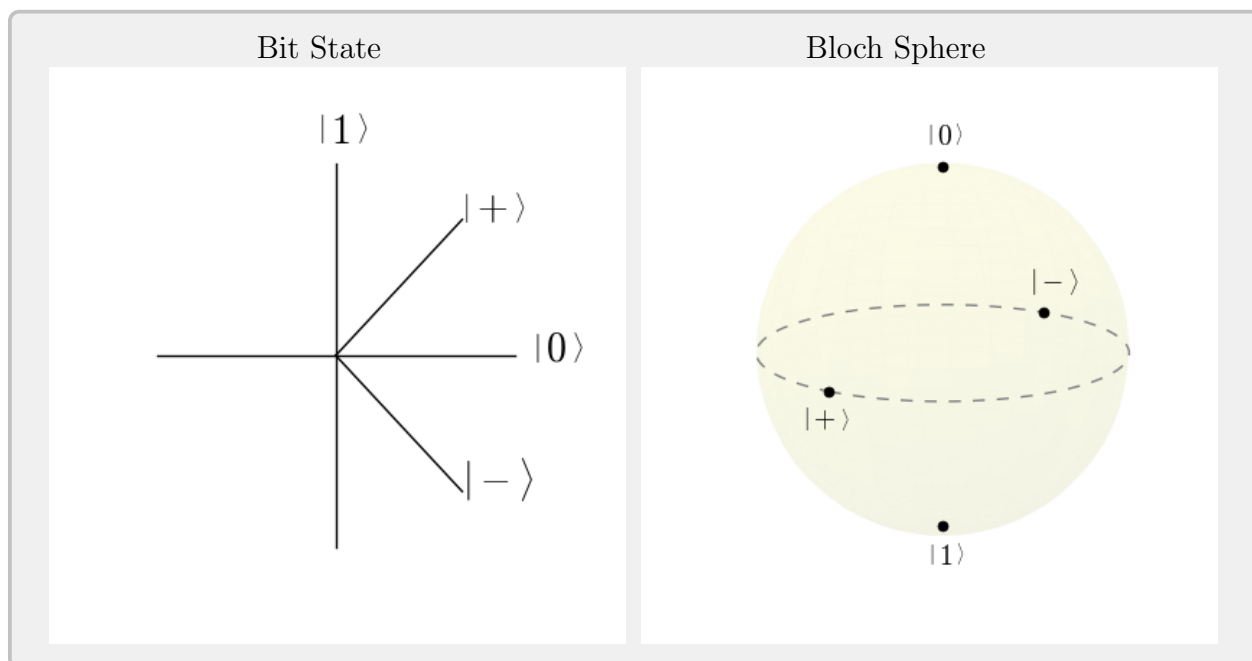


The Bloch sphere is typically used in the quantum information context. So rather than talk about spin vectors, we'll often be talking about the states of quantum bits. All the math is completely equivalent. Just remember that:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = | + z \rangle = | 0 \rangle, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = | - z \rangle = | 1 \rangle$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = | + x \rangle = | + \rangle, \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = | - x \rangle = | - \rangle$$

Also, where before we thought of the spin space as the actual three dimensional physical space in which we might (for example) do a Stern-Gerlach type experiment, now we're thinking of the Bloch sphere as another abstract type of mathematical space in which we'll visualize various aspects of quantum bits. So we may have something more like this:



Here I've just plotted points on the sphere to represent the various bit states, rather than arrows. Note that all valid (pure) quantum bit states will appear on the *surface* of the Bloch sphere. That is, they will be at a distance of 1 from the origin. If we plot them as vectors, they will have a length of 1.

## 2 Translating between the two spaces

Translating between the quantum “state space” (the complex vector space) and the “spin space” (the space of physical spin vectors or the Bloch sphere) can be summarized as follows:

To go from spin space to state space:

1. Determine the components of the spin vector.
2. Use them to construct a weighted sum of  $\sigma_x$ ,  $\sigma_y$ , and  $\sigma_z$ .
3. The resulting matrix represents the quantum system in state space.

To go from state space to spin space:

1. Create a matrix representing the quantum system.
2. Decompose it as a weighted sum of  $\sigma_x$ ,  $\sigma_y$ , and  $\sigma_z$ .
3. Use the weights as the elements of the spin vector.

## 2.1 Going from Spin Space to Vector Space

Given a spin vector  $\vec{S}$  we want to determine the corresponding bit state  $|S\rangle$ . Here are the steps in detail:

- (1) Determine the components of the spin vector

Given the spin vector  $\vec{S}$ , you want the vector's three components  $S_x, S_y$ , and  $S_z$ . If you were given  $\vec{S}$  in vector form then you already have them. If you were given  $\vec{S}$  in some other form (based on angles, for example) then  $S_x, S_y$ , and  $S_z$  are the projections of  $\vec{S}$  onto the  $X, Y$ , and  $Z$  axes.

- (2) Use them to construct a weighted sum of  $\sigma_x, \sigma_y$ , and  $\sigma_z$ .

The three Pauli matrices are the observable operators which represent the  $X, Y$ , and  $Z$  bases. Write out a linear combination of them weighted by  $S_x, S_y$ , and  $S_z$  like this:

$$\sigma_S = S_x\sigma_x + S_y\sigma_y + S_z\sigma_z = S_x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + S_y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + S_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- (3) The resulting matrix represents the quantum system in state space

The result of the sum above is  $\sigma_S$ , which is the observable matrix for the  $S$  basis. That is, the basis consisting of the states  $|+S\rangle$  and  $|-S\rangle$ .

To determine the actual state, find the eigenvectors of  $\sigma_S$ . The vector with the +1 eigenvalue is  $|+S\rangle$  or simply  $|S\rangle$ , the state corresponding to the spin vector  $\vec{S}$ .

The procedure for finding the eigenvalues and eigenvectors of a 2x2 matrix can be found in section 5. But the eigenvectors won't always come out nicely. I would encourage you to simply use the computer to find them. You can use the Python code I supplied last year, or there are a couple of websites that will do it which are also listed in section 5.

**Example:**  $\vec{S}$  is given as  $30^\circ$  up from the  $X$  axis

Suppose we're given the spin vector in angle form as being  $30^\circ$  up from the  $X$  axis.

Step (1):

The projection along the  $X$  axis is  $\cos(30^\circ) = \sqrt{3}/2$ . Along the  $Z$  axis its  $\sin(30^\circ) = 1/2$ . Since  $\vec{S}$  is in the  $XZ$  plane there is no projection onto  $Y$ . So we have:

$$S_x = \frac{\sqrt{3}}{2}, \quad S_y = 0, \quad S_z = \frac{1}{2}.$$

Step (2):

The weighted sum of the sigmas is:

$$\sigma_S = \frac{\sqrt{3}}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$$

Step (3):

From the observable matrix  $\sigma_S$  we can calculate the eigenvectors (and states) as:

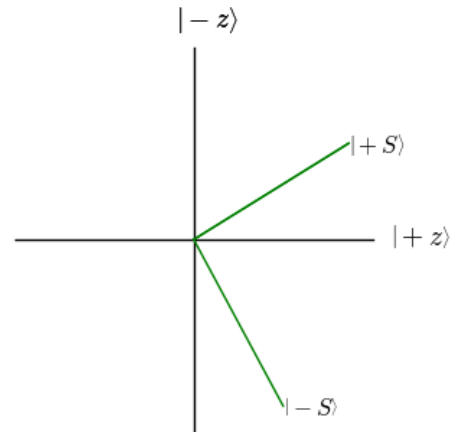
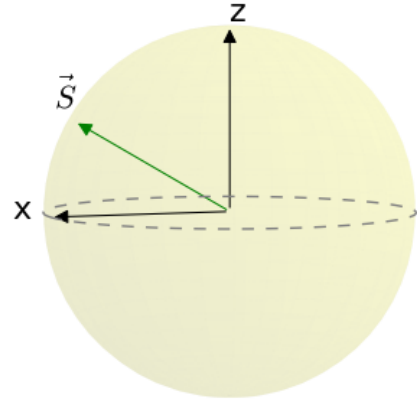
$$|+S\rangle = \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \quad |-S\rangle = \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$$

(See section 5 for the details.)

You can verify that these states *are* the eigenvectors of  $\sigma_S$  by doing the multiplications:

$$\frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix} = (+1) \frac{1}{2} \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}, \quad \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix} = (-1) \frac{1}{2} \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$$

Since the states we found don't contain any imaginary parts, we can plot them on the 2-D state space plane (above). Note that  $|+S\rangle$  and  $|-S\rangle$  are orthogonal, as they must be in order to form an orthonormal basis.



## 2.2 Going from Vector Space to Spin Space

Now we'll look at the detailed steps in going from state space to spin space. Step 1, creating the matrix, may be done in different ways depending on the situation. But the other two steps will be the same. So we'll cover steps 2 and 3 first and then talk about step 1.

(1) Create a matrix representing the quantum system.

We're just going to assume for the moment you have the matrix. For example, this could be an observable matrix, such as the  $\sigma_s$  we constructed in the example above.

(2) Decompose it as a weighted sum of  $\sigma_x$ ,  $\sigma_y$ , and  $\sigma_z$ .

Any valid matrix which represents a state is going to be in the form:

$$\begin{pmatrix} z & x - yi \\ x + yi & -z \end{pmatrix}$$

Therefore the decomposition is easy. The values  $x, y$ , and  $z$  in the matrix *are* the weights of  $\sigma_x, \sigma_y$ , and  $\sigma_z$ :

$$\begin{pmatrix} z & x + yi \\ x - yi & -z \end{pmatrix} = x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(3) Use the weights as the elements of the spin vector.

So the desired spin vector (Bloch vector) is simply:  $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Now we'll talk about how to create the matrix that represents the state. There are two kinds of states we wish to represent, pure states and mixed states. When working with a pure state, we might start out with either the state vector or the state's density matrix. When working with a mixed state, there is no state vector so we'll definitely be using a density matrix. If we're working with one of the bits derived from an entangled state, then we treat it as mixed and it's represented by a density matrix.

When we plot a pure state it will be a vector of length one (or if depicted as a dot it will appear on the surface of the Bloch sphere). When we plot a mixed state (or a bit from an entangled state) it will be a vector with length less than one (or the dot will appear on the inside of the sphere). A maximally mixed (or entangled) bit will be a dot at the center of the sphere.



### 2.3 Creating the observable matrix from a state vector

Suppose we have a state  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ .

Then the matrix we are looking for is the observable matrix for the basis in which  $|\psi\rangle$  is the “+” eigenvector state.

In general, given a basis  $\{|b_n\rangle\}$ , the observable matrix for that basis is:

$$\sum_n \lambda_n |b_n\rangle\langle b_n|, \text{ where } \lambda_n \text{ is the eigenvalue associated with the } n\text{th basis vector.}$$

In other words, you can find the observable for a basis by adding up the outer products of the basis vectors weighted by their desired eigenvalues.

So here we have the state  $|\psi\rangle$  and we want it to have the +1 eigenvalue. Since this is a one-bit state, we will have one additional basis vector, let's call it:  $|\psi^\perp\rangle$ , which we want to have the -1 eigenvalue. So the summation will be:

$$|\psi\rangle\langle\psi| - |\psi^\perp\rangle\langle\psi^\perp|$$

What is  $|\psi^\perp\rangle$  (the complementary or orthogonal state to  $|\psi\rangle$ )? Well, if  $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ , then  $|\psi^\perp\rangle$  is  $\beta^*|0\rangle - \alpha^*|1\rangle$ . If  $\alpha$  and  $\beta$  are real, then you should be able to see, simply by looking at their angles in state space, that they're orthogonal. Algebraically:

$$\langle\psi|\psi^\perp\rangle = \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix} \begin{pmatrix} \beta^* \\ -\alpha^* \end{pmatrix} = \alpha^*\beta^* - \alpha^*\beta^* = 0$$

Example: Given the state  $|\psi\rangle = \frac{1}{3}|0\rangle + \frac{\sqrt{8}}{3}|1\rangle$ , then  $|\psi^\perp\rangle = \frac{\sqrt{8}}{3}|0\rangle - \frac{1}{3}|1\rangle$

(1) Create the matrix

$$\sigma_\psi = |\psi\rangle\langle\psi| - |\psi^\perp\rangle\langle\psi^\perp| = \begin{pmatrix} \frac{1}{3} \\ \frac{\sqrt{8}}{3} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{\sqrt{8}}{3} \end{pmatrix} - \begin{pmatrix} \frac{\sqrt{8}}{3} \\ -\frac{1}{3} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{8}}{3} & -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} -\frac{7}{9} & \frac{4\sqrt{2}}{9} \\ \frac{4\sqrt{2}}{9} & \frac{7}{9} \end{pmatrix}$$

(2) Decompose it  $\begin{pmatrix} -\frac{7}{9} & \frac{4\sqrt{2}}{9} \\ \frac{4\sqrt{2}}{9} & \frac{7}{9} \end{pmatrix} = \frac{4\sqrt{2}}{9} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{7}{9} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(1) And the spin vector is:  $\begin{pmatrix} \frac{4\sqrt{2}}{9} & 0 & -\frac{7}{9} \end{pmatrix}$

## 2.4 Creating the observable matrix from a density matrix

There's another way to get the observable matrix for a one-bit state, and that is to work with the state's density matrix.

When you add up the outer products of all the vectors in a basis, you get the identity operator:

$$\sum_n |b_n\rangle\langle b_n| = I$$

This is referred to as the “completeness relation” (i.e., if this formula holds then you have a complete set of basis vectors).

Bearing in mind (from above) that the observable for the state  $|\psi\rangle$ , which we call  $\sigma_\psi$  is

$$|\psi\rangle\langle\psi| - |\psi^\perp\rangle\langle\psi^\perp|$$

We can derive the formula:

$$\sigma_\psi = 2\rho_\psi - I$$

(Where  $\rho_\psi$  is the density matrix for  $|\psi\rangle$ .)

The algebra goes like this:

$$\begin{aligned}\sigma_\psi &= 2\rho_\psi - I \\ &= 2|\psi\rangle\langle\psi| - I \quad | \text{Because } \rho_\psi \text{ is the outer product of } |\psi\rangle \text{ with itself.} \\ &= 2|\psi\rangle\langle\psi| - (|\psi\rangle\langle\psi| + |\psi^\perp\rangle\langle\psi^\perp|) \quad | \text{Replace } I \text{ with the sum of the basis vectors.} \\ &= |\psi\rangle\langle\psi| - |\psi^\perp\rangle\langle\psi^\perp| \quad | \text{Do the subtraction.}\end{aligned}$$

One big advantage of this method is that you can use it for *any* density matrix. So it can also be used to plot mixed states on the Bloch sphere. When you start out with a mixed state, then the  $\sigma$  you obtain is not literally an observable operator, but it's still the matrix that you need to decompose in order to get the spin vector.

When you plot a mixed state in spin space, you get a vector having a length of less than one (or a point on the inside of the unit sphere).

### 3 Some Examples

Example-1

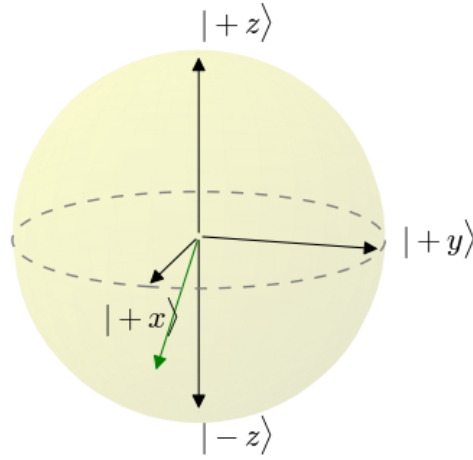
$$|\psi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$$

$$\rho = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}$$

$$\theta = \frac{2\pi}{3} \approx 2.09$$

$$\phi = 0 \approx 0.00$$

$$\vec{S} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}, \quad |\vec{S}| = 1$$



$$|\psi\rangle = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \quad \rho = \begin{bmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{3}{4} \end{bmatrix}, \quad \sigma_\psi = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \left(\frac{\sqrt{3}}{2}\right) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \left(-\frac{1}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Example-2

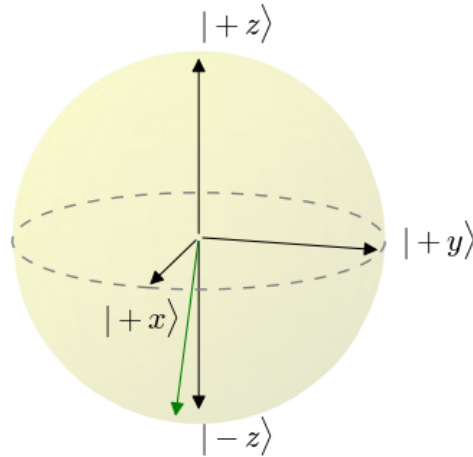
$$|\psi\rangle = \frac{1}{4}|0\rangle + \frac{\sqrt{15}}{4}|1\rangle$$

$$\rho = \begin{bmatrix} \frac{1}{16} & \frac{\sqrt{15}}{16} \\ \frac{\sqrt{15}}{16} & \frac{15}{16} \end{bmatrix}$$

$$\theta = \arccos\left(-\frac{7}{8}\right) \approx 2.64$$

$$\phi = 0 \approx 0.00$$

$$\vec{S} = \begin{bmatrix} \frac{\sqrt{15}}{8} \\ 0 \\ -\frac{7}{8} \end{bmatrix}, \quad |\vec{S}| = 1$$



$$|\psi\rangle = \begin{bmatrix} \frac{1}{4} \\ \frac{\sqrt{15}}{4} \end{bmatrix}, \quad \rho = \begin{bmatrix} \frac{1}{16} & \frac{\sqrt{15}}{16} \\ \frac{\sqrt{15}}{16} & \frac{15}{16} \end{bmatrix}, \quad \sigma_\psi = \begin{bmatrix} -\frac{7}{8} & \frac{\sqrt{15}}{8} \\ \frac{\sqrt{15}}{8} & \frac{7}{8} \end{bmatrix} = \left(\frac{\sqrt{15}}{8}\right) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \left(-\frac{7}{8}\right) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

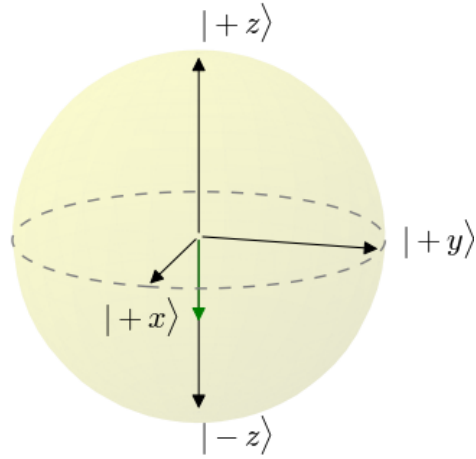
### Example-3

$$\rho = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{bmatrix}$$

$$\theta = \pi \approx 3.14$$

$$\phi = 0 \approx 0.00$$

$$\vec{S} = \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix}, \quad |\vec{S}| = \frac{1}{2}$$



$$|\psi\rangle = \text{None}, \quad \rho = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{3}{4} \end{bmatrix}, \quad \sigma_\psi = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} = \left(-\frac{1}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

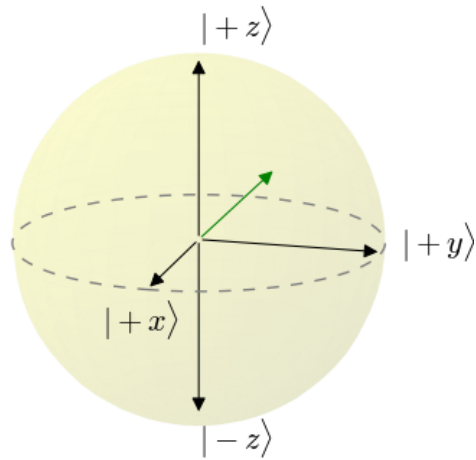
### Example-4

$$\rho = \begin{bmatrix} \frac{3}{4} & \frac{1}{6} - \frac{i}{4} \\ \frac{1}{6} + \frac{i}{4} & \frac{5}{12} \end{bmatrix}$$

$$\theta = \arccos\left(\frac{3\sqrt{22}}{22}\right) \approx 0.88$$

$$\phi = \arctan\left(\frac{3}{2}\right) \approx 0.98$$

$$\vec{S} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \quad |\vec{S}| = \frac{\sqrt{22}}{6}$$



$$|\psi\rangle = \text{None}, \quad \rho = \begin{bmatrix} \frac{3}{4} & \frac{1}{6} - \frac{i}{4} \\ \frac{1}{6} + \frac{i}{4} & \frac{5}{12} \end{bmatrix}, \quad \sigma = \begin{bmatrix} \frac{1}{2} & \frac{1}{3} - \frac{i}{2} \\ \frac{1}{3} + \frac{i}{2} & -\frac{1}{6} \end{bmatrix} = \left(\frac{1}{3}\right) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \left(\frac{1}{2}\right) \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + \left(\frac{1}{2}\right) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

## 4 Summary of Formulae

### 4.1 The Completeness Relation

Given an (orthonormal) set of basis vectors:  $\{|b_n\rangle\}$ , the sum of their outer products will add up to the identity operator:

$$\sum_n |b_n\rangle\langle b_n| = I$$

### 4.2 Constructing an observable matrix (general case)

Given the basis vectors:  $\{|b_n\rangle\}$  and the associated eigenvalues:  $\{\lambda_n\}$ , the observable matrix for the basis (sometimes called  $\sigma_b$ ) is:

$$\sum_n \lambda_n |b_n\rangle\langle b_n|$$

### 4.3 Constructing an 1-bit “observable” matrix (including mixed states)

If you only have a 1-bit state  $|\psi\rangle$  you can also do the following:

- (1) Create the density matrix  $\rho = |\psi\rangle\langle\psi|$ .
- (2)  $\sigma_\psi = 2\rho - I$ .

If you already have the density matrix (say for a mixed state) you can just do:

$$\sigma_{mix} = 2\rho - I.$$

### 4.4 Decomposing the observable matrix

$$\begin{pmatrix} z & x - yi \\ x + yi & -z \end{pmatrix} = x \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + y \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

### 4.5 To get the angles of the spin vector from the state

Given a state  $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$  you can get the angles of the spin vector  $\vec{S}$  as:

$$\theta = 2 \cos^{-1}(|\alpha|), \text{ where } \theta \text{ is the angle from } \vec{Z} \text{ to } \vec{S}.$$

$$\phi = \arg(\beta) - \arg(\alpha), \text{ where } \phi \text{ is the angle from } \vec{X} \text{ to } \vec{S}.$$

Note that  $\arg()$  gives the angle of  $\alpha$  or  $\beta$  on the complex plane.

## 5 Finding Eigenvalues and Eigenvectors

### 5.1 Finding eigenvectors on the computer

This is the way I recommend doing it. In most cases (other than carefully constructed examples) the eigenvectors you find are going to be complicated (roots within roots for example) and will be better done as numeric approximations. Also, there is not a lot to be “learned” by doing this process by hand. Of course, if you’re about to take a linear algebra class then you had better learn to do it :-)

If you’re using the Python code I distributed earlier, see the notebook: **0100\_How\_to\_use\_sglib.ipynb** and look for the section called “Finding Eigenvectors.”

Otherwise, here are two websites that will do it:

**http://www.wolframalpha.com** will accept a query like this:

```
eigenvalues {{0,1,2,0},{1,0,1,1},{1,1,1,1},{0,0,0,1}}
```

**http://www.sympygamma.com** will take something like this:

```
Matrix([(0,1,2,0),(1,0,1,1),(1,1,1,1),(0,0,0,1)])
```

For the example in section 2.1, with  $\sigma_S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}$

You can calculate the eigenvectors on Sympy Gamma using:

```
Matrix([(1/2, sqrt(3)/2), (sqrt(3)/2, -1/2)])
```

or on Wolfram Alpha using:

```
eigenvalues {{1/2, sqrt(3)/2}, {sqrt(3)/2, -1/2}}
```

Remember that the vectors you get from these websites still need to be normalized.

## 5.2 Finding eigenvectors by hand

Given a matrix  $M$  and a vector  $|v\rangle$ , if  $M|v\rangle = \lambda|v\rangle$  we say that  $|v\rangle$  is an eigenvector of  $M$  and  $\lambda$  is the eigenvalue associated with  $|v\rangle$ . In the situations we will deal with it is typical for a matrix of dimension  $n$  to have  $n$  eigenvectors, each with an associated eigenvalue.

Given a matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  and a vector  $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$  some algebra will show that, for the vector

to be an eigenvector of the matrix, the eigenvalue  $\lambda$  must satisfy the 2nd degree polynomial equation:

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0$$

This is sometimes called the *characteristic equation* of the matrix. This equation can always be solved, either by factoring or by the quadratic formula:

$$\lambda = \frac{1}{2} \left( -B \pm \sqrt{B^2 - 4C} \right), \quad B = -(a + d), \quad C = (ad - bc)$$

But the fact that this equation can always be solved for  $\lambda$  does not mean that every  $2 \times 2$  matrix has valid eigenvectors. For example, it will sometimes turn out either that  $\lambda$  is zero, or that the resulting eigenvectors are filled with zeros. In cases like this, we do not have valid eigenvectors.

Once you have found the possible  $\lambda$ s (there will typically be two different ones, for a dimension 2 matrix) then you take each one and plug it back in to the original eigenvalue equation. What you will end up with is simply a *ratio between the two components of the vector*. For the matrix and vector above, you will get:

$$\alpha = \frac{-b\beta}{a - \lambda}$$

For matrices with dimension of  $n$  greater than 2, the same principle still works, but the characteristic equation will be of the  $n$ th degree, and may not be solvable. Normally we would go to the computer to find eigenvalues if we're dealing with more than a 'one bit' state.

## 6 Unitary operations as rotations on the Bloch sphere

(TBD)