Special Relativity #1

This is essentially a set of notes from some previous meetings of the old Portland group. This may not be exactly the material we're covering in the current series, but there should be some overlap.

Relativity before and after Einstein

Very roughly speaking ...

Before Einstein: Motion is relative. Time and space are absolute.
After Einstein: The speed of light is absolute. Time and space are relative.

Today we'll talk about relativity before Einstein (Galilean relativity). Then, for several weeks, we'll talk about what Einstein did (Special Relativity).

Galilean relativity

Galileo Galilei was born around the middle of the 16th century. He was one of the first people who practiced what we now call "physics," although I imagine that he probably thought of himself as a "natural philosopher." From his writings it was clear that he totally understood the relativity of motion, and so the fact that "motion is relative" is often referred to as "Galilean relativity." Galileo died the same year that Isaac Newton was born. By Newton's time everyone who studied such things was well aware of this kind of "relativity."

Inertial motion = constant velocity

Now, this does *not* mean that *all* motion is relative. What we are talking about here is what is called *inertial motion*. Inertial motion simply means motion with a constant velocity.

The term velocity includes both speed and direction. So if you are travelling one kilometer per hour east, this is a *different velocity* than if you were travelling one km/hour west. So, constant velocity also implies that you are traveling in a straight line. If you're going in a circle, for example, your velocity is changing at every moment (because your direction is changing).

Another way to define inertial or constant velocity is to say that it is *unaccelerated* motion. If a force is applied to an object, then the object accelerates, and – as long as the force is being applied – the motion is not constant.

So, before Einstein came along, "relativity" meant that:

Interial motion is relative to the reference frame of the observer.

What's a "reference frame?" Your reference frame is simply your point of view. How things look relative to you. So, for example, in your reference frame you are always standing still. Other things may or may not be moving.

This is, of course, assuming that you are not accelerating. If you're accelerating you can *feel* it. Picture that you're in a box out in space somewhere. There is no experiment you could do inside that box that would tell you whether or not you were "moving" (at a constant velocity). If you accelerate, however, you can easily tell.

So, one of the main points about Galilean relativity and inertial motion is that *there is no possible experiment that can be performed inside a reference frame that can tell if the frame is "moving" or not.* Essentially, there is no "matter of fact" about whether a reference frame (or an object) is "moving" or "standing still." These terms really only have meaning *relative* to some other frame (or some other object).

But *acceleration* of a frame (or object) is very real, and is *not* relative.

Example of the usefulness of relativity of motion

(Note: This example is straight from chapter one of Mermin's book: It's About Time.)

It would be nice to have a picture here. Maybe somebody will draw one and scan it in. In any event, here are some cryptic notes that will hopefully help recall what we had up on the board.

Scenario 1

Before: $- - - > \bullet\bullet < - - -$
After: $\bullet < - - - - - - - - > \bullet$ (Obvious?)

Scenario 2

Before: $- - - > \bullet\bullet$
After: (not so obvious?)

Conservation of momentum says: before the collision $p = mv$. After the collision you could have them both moving to the right at $\frac{1}{2}v$, or any number of other combinations as long as the total $p = mv$.

But conservation of (kinetic) energy says that before the collision $K = \frac{1}{2}mv^2$. So, after the collision we have to have *both* $mv_1 + mv_2 = mv$ and $\frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 = \frac{1}{2}mv^2$.

The only simultaneous solution to both of these two equations (that makes sense physically) is: $v_2 = v$. The right-moving mass stops, and the previously stationary mass moves right with the original velocity of the other mass.

Simpler solution using relativity

But we can find the solution in a much simpler way using Galilean relativity. Change your frame of reference so that you are looking at the problem from the point of view of an observer who is travelling to the right at $(1/2)v$. Now we have exactly Scenario 1 (except that the masses are travelling at $\pm(1/2)v$ instead of $\pm v$). Clearly, after the collision (in our new frame of reference) the masses are travelling apart at $(1/2)v$. Finally, go back to the original frame and we have the left-hand mass standing still and the right-hand mass travelling right with velocity v. There is no need to invoke 2 different conservation laws, or do any math other than one addition and one subtraction.

The situation at the the beginning of the 20th century

With the exception of gravity, you might argue that "most of physics" is ultimately explained by:

(1) Newton's 'laws of motion' and in particular the '2nd law'

$$F = ma$$

which is compatible with Galilean relativity, and

(2) Maxwell's Equations

$$\nabla \bullet \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \bullet \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{\mathbf{j}}$$

which predict the speed of light, but have no obvious frame of reference.

The Ether hypothesis attempts to establish a universal frame of reference for the propagation of EM waves, but has many philisophical and practical problems. The Michelson-Morley experiment in 1887 fails to find any difference in the speed of light, in what ought to be different reference frames.

This problem with the speed of light was, in retrospect, one of the two "great" unsolved problems of physics, around the turn of the 20th century. The other one being the so-called ultraviolet catastrophe.

Simplifying things a bit: the problems surrounding the speed of light led to Einstein's relativity and the ultraviolet catastrophe led to quantum mechanics.

Special Relativity #2

What's "special" about Special Relativity

Einstein first came up with a "theory of relativity" that solved the "speed of light problem" and modified Newton's theory of motion. But Newton's theory of gravity couldn't be accommodated in the theory. It took Einstein about 10 more years to address gravity. The result was General Relativity.

The original theory was then called "special" relativity because it was only a special (no gravity) case of the more general theory.

Special relativity is very simple mathematically, even though the consequences of the theory imply a dramatic change to our notions of space and time. Special relativity could be (and perhaps is?) easily taught in a high school physics class. General relativity is *much* more complicated mathematically, and is typically taught to university physics majors only at a graduate level (if at all).

The postulates of special relativity

1. The speed of light is the same in all inertial frames of reference.
2. There is no experiment that can be done, internal to a given inertial frame of reference, that can distinguish that frame from any other frame.

Or to say it another way:

1. The speed of light is absolute.
2. But, Galilean relativity still holds!

The implications of a fixed speed of light

How is it possible to keep the speed of light fixed? We consider two observers in two different reference frames, who watch a photon (a "particle" of light) bounce off a mirror. Even though the two frames are moving relative to each other, we simply *assume* that both observers see the photon going the same speed. This does *not* produce a logical contradiction, as long as we are willing to concede that *the two observers experience a different duration of time* during the journey of the photon. This is called *time dilation.*

It is important to understand that this is not some sort of illusion, or subjective experience. The duration of time in question is actually different in the two different reference frames.

With a similar argument we showed that lengths in space are different for two different observers. This is known as *length contraction.*

I don't have the pictures we drew on the board, but I've tried to document the sequence of steps so that you can (hopefully) reproduce the calculations at home, if you like.

But here's the key point: *Durations of time and lengths of space are, in a sense, like projections of vectors along the axes of an arbitrary coordinate system. You can change their relative magnitude by changing the coordinates.*

Time Dilation

1. Draw the two reference frames, Alice has the mirror. Bob and Alice both agree on: c (the speed of light), v (the magnitude of their relative velocity), and h (the height of the mirror).

2. Alice bounces a photon of light off the mirror. She sees the photon's path as having a length of $2h$ traveled in time t_a. In other words, Alice measures the elapsed time as: $t_a = \frac{2h}{c}$.

3. Bob measures time t_b for the photon go from Alice to the mirror and back. But during that time he sees Alice's entire frame of reference move toward the right at velocity v. So he sees an equilateral triangle with a base of length vt_b.

4. Forming a right triangle from half of that base, with the height h as the other side, then during time $t_b/2$ Bob sees the photon travel the hypotenuse, which has length $\sqrt{h^2 + \frac{1}{4}v^2t_b^2}$.

5. Given that Bob saw the photon travel that same hypotenuse in time $t_b/2$, we conclude that $\frac{1}{2}ct_b = \sqrt{h^2 + \frac{1}{4}v^2t_b^2}$.

6. Solving for t_b

   $\frac{1}{4}c^2t_b^2 = h^2 + \frac{1}{4}v^2t_b^2$ (square both sides)

   $\frac{1}{4}c^2t_b^2 - \frac{1}{4}v^2t_b^2 = h^2$ (subtract term)

   $\frac{1}{4}t_b^2(c^2 - v^2) = h^2$ (factor left side)

   $t_b^2 = \frac{4h^2}{c^2 - v^2}$ (divide)

   $t_b = \frac{2h}{\sqrt{c^2 - v^2}}$ (take positive square root)

7. Since Alice was present at both "events" (explain) her time measurement is considered the *proper time*, noted as t_0, between those events.

8. We are looking for a conversion factor between the proper time and Bob's time: $t_0 \times ? = t_b$. That is:

   $\frac{2h}{c} \times ? = \frac{2h}{\sqrt{c^2 - v^2}}$

9. We can cancel the $2h$ from both sides, and then clearly: $\frac{1}{c}\left(\frac{c}{1}\frac{1}{\sqrt{c^2 - v^2}}\right) = \frac{1}{\sqrt{c^2 - v^2}}$

   Simplifying, we have the conversion factor: $\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

   And the general formula relating a proper time t_0 to the time t of any observer who does not see the proper time.

   $t = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}t_0$

10. What we can conclude from this is that the proper time is always the the shortest time observed. Any other time will appear to be longer (and hence "dilated").

Length Contraction

1. Now put two observers on either end of the ship and one on the ground.
2. The observers on the ship each note the time that they pass the observer on the ground and they calculate t_s as the difference between the two.
3. The observer on the ground notes the time between the passage of the two ends of the ship and calls that t_g.
4. The "events" here are the times that each end of the ship passed the observer on the ground. So that observer sees the proper time: $t_g = t_0$.
5. Hence $t_s = \frac{t_g}{\sqrt{1-\frac{v^2}{c^2}}}$.
6. The observers on the ship calculate $L_s = v \times t_s$ an the one on the ground calculates $L_g = v \times t_g$.
7. You might think that L_g would be the "proper" length because it was calculated with the proper time, but no. The *proper length* is defined as the length of the object measured in *that object's own frame of reference* (which makes sense when you think about it, since this is the only way of defining it uniquely).
8. We want to find the multiplier which will take us from the proper lentgh $L_0 = L_s$ to the other length L_g.
9. So we want: $v\frac{t_g}{\sqrt{1-\frac{v^2}{c^2}}} \times (?) = vt_g$
10. Cancel vt_g from both sides and it's obvious that we have to multiply by $\sqrt{1-\frac{v^2}{c^2}}$.
11. So: $L = L_0\sqrt{1-\frac{v^2}{c^2}}$.

Summary

1. The *proper time between two events*, t_0, is the time measured by an observer who is present at both events.
2. The *proper length of an object*, L_0, is the length as measured in the object's own frame of reference.
3. $t = \frac{t_0}{\sqrt{1-\frac{v^2}{c^2}}}$.
4. $L = L_0\sqrt{1-\frac{v^2}{c^2}}$.

Special Relativity #3

Invariance and the spacetime "interval"

I'm going to shift gears now for a bit, and talk about something called *invariance*. Invariance is the property of not changing when subjected to some specific set of operations.

As a first example, consider a vector in 2 dimensional Euclidian space (an arrow on the whiteboard). If I draw an xy coordinate system, then I can represent the arrow as a column vector. Perhaps the head of the arrow is at the point x=3 and y=2. Then the components of the column vector will be 3 and 2.

If I change the coordinate system (rotate the xy axes by a few degrees) then the components of the column vector will change. The components of a vector are not invariant. But the length of the vector will never change no matter how much I rotate the axes. The length doesn't *vary* and so it is called invariant.

We would say something like: *the length of a vector in Euclidean space is invariant under rotations of the coordinate system.*

This is more than a mathematical nicety. The implication is that the length of a vector is a *real* thing, whereas the components in the vector's column representation are merely *artifacts* of some specific choice of coordinates.

Similarly, in a complex vector space (the space of quantum bits) the length of a vector $|v\rangle$ is $\sqrt{\langle v|v\rangle}$. $|v\rangle$ can be expressed in any number of different bases, but no matter what basis we select the length will not change. Change of basis is a little bit more general than simple rotation, but a vector's length is still invariant under change of basis. A change of basis is also a change of coordinate systems.

Now back to special relativity. Before Einstein, we thought that we lived in 3 dimensional Euclidean space. But we have now seen that lengths in space (and in time!) *change* based on the reference frame of the observer. In other words:

1. A particular observer's frame of reference is a coordinate system.
2. When we change from one observer's viewpoint to another we are changing the coordinate system (like a change of basis).
3. Extensions in space and time are merely artifacts of the specific coordinate system chosen.
4. We can conclude that we don't actually live in 3D euclidean space!

If the extensions in time and space are not "real" then what is real? It turns out that if you take all the space extensions and add them up as if you were going to calculate the distance in euclidean space, and then *subtract* the time extension, you will get an invariant quantitiy:

$$s = \sqrt{x^2 + y^2 + z^2 - t^2}$$

If we don't live in euclidean space then where do we live? Not long after Einstein published his first paper on what we now know as special relativity, Hermann Minkowski observed that the theory had a startling implication. That we actually live in a four-dimensional world of space-time (often called Minkowski spacetime or simply Minkowski space).

The invariant s above is called the *spacetime interval* and you could (perhaps) think of it as the true *distance* in four dimensional spacetime.

It is very important to understand the difference between spacetime and 4D euclidean space. Suppose we simply had a "fourth dimension" of space. This would mean that we would be able to find *four* angles that were all mutually orthogonal (all 90° from each other). If we called the fourth direction w then the distance in this space would be given by the formula:

$$D = \sqrt{x^2 + y^2 + z^2 + w^2}$$

The minus sign in the formula for the space time interval makes it clear that time is not just "the fourth dimension" that is portrayed in sci-fi movies. Time is defintely not at a 90° angle from space. It's important to keep this in mind, since almost every diagram of space-time (including the ones that we are going to draw) make it *appear* as if time *is* at a 90° angle.

What "direction" *does* time go? I have no idea.