The Implicit Summation Convention

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Brief review of summation notation

I'm assuming everybody pretty much understands summation notation, but let's just have a brief review by looking at a few examples:

$$(1)\ \ \sum_{n=1}^{5} n$$

We just added up the numbers 1 through 5.

$$(2)\ \ \sum_{k=0}^{3} M_{kk}$$

Here we computed the trace of a 3x3 matrix M. (We added up the main diagonal.)

$$(3)\ \ \sum_{jk=0}^{3} M_{jk}$$

Each of the two index variables *individually* goes from 0 through 3.
This works out to: $M_{00} + M_{01} + M_{02} + M_{10} + M_{11} + M_{12} + M_{20} + M_{21} + M_{22}$.
In other words, we added up every element of the matrix.

Often, when we already know the range of the numbers involved, we leave out the range specification in the summation. In the case above, if the dimension of the matrix is already given, we would just write: $\displaystyle\sum_{jk} M_{jk}$

The next two examples are especially pertinent to our explanation of "implicit summation."

Here is the dot (inner) product of two (real) vectors v and w:

$$(4)\ \ v \bullet w = \sum_{k} v_k w_k$$

And here is a matrix M multiplying a vector v, giving a new vector w:

$$(5)\ \ w_j = \sum_{k} M_{jk} v_k$$

Note that in this last example the expression literally just computes one component of the vector w, the jth component. But if we know one arbitrary component then we know the whole vector. *Please work this one out, with at least a 2D matrix and vector.*

Some words about indices

In GR we are going to use both upper and lower indices. In other words, an index can be either a subscript or a superscript. So we can have either v_k and M_{jk} or v^k and M^{jk}. We'll learn more about what upper and lower indices *mean* later. But for now we just need to know that they exist.

Also in GR Greek letters are commonly used for indices, particularly μ and ν. We might as well start getting used to those now. So we'll write v_ν and $M_{\mu\nu}$ or v^ν and $M^{\mu\nu}$.

(And finally) Implicit summation

In GR there is an implicit summation convention, sometimes called the "Einstein summation convention." The basic rule is this:

> *When you have matching upper and lower indices you sum over them.*

This has a number of implications. For example, there is nowhere to put the range being summed over. So this only works when you know the dimension of the space you're working in. For us this will typically be 4D space time with indices running from 0 through 3. We'll deal with other implications of the notation as they arise.

Here are examples (4) and (5) from above, modified to use Greek indices and implicit summation:

Below is the inner product of the vectors v and w. We are implicitly summing over μ.

$$(4')\quad v \bullet w = v_\mu w^\mu$$

Next we'll multiply the vector v by the matrix M to give the new vector w. We are implicitly summing over ν. The μ index specifies the single row of M that produces the single μth component of w. Given an expression for the μth component, we know all the components.

$$(5')\quad w_\mu = M_{\mu\nu} v^\nu$$

Note that with the summation symbol gone, it kind of *looks* like we're just multiplying a matrix times a vector: $\vec{w} = M\vec{v}$. In fact, the result is the same. I'll say more about this similarity in the notation later.