First look at Tensors

April 2, 2019

What is a Tensor?

(1) It's one of the objects we already use:

A rank 0 tensor is a scalar or number
A rank 1 tensor is a vector or list of numbers
A rank 2 tensor is a matrix or list of lists of numbers
A rank 3 tensor is a list of lists of lists of numbers
and so on ...

We'll encounter tensors up through rank 4 in general relativity.

Note that the *dimension* of a tensor relates to the number of components it has, and is what we normally think of as the dimension of the space we're working in. The *rank* of a tensor is the number of indices it has.

(2) To be a tensor, the object must transform "properly" with respect to the mathematical space you're working in. More on this later, but note that the "four-vectors" we talked about earlier are rank 1 tensors (in Minkowski space).

Spaces, Inner Products, and Metrics

Here are some "mathematical spaces" that we have previously worked in:

$\mathbb{C}^n$ - the space of vectors with complex components used for quantum information.
L^2 - the space of functions with complex arguments used for quantum mechanics.
E^3 - three dimensional Euclidean space used for Newtonian physics.
M^4 - four dimensional Minkowski space used for special relativity.

Each of these spaces has an *inner product* and a *metric*. The metric is in some sense a measure of distance in that space. You may think of it as the distance between two points, the length of a vector, or (in spacetime) the spacetime interval. The metric can be calculated by using the inner product. In fact, if things are set up properly, the metric in all of these spaces can be written as $\sqrt{\langle v|v\rangle}$, where v is a valid rank 1 tensor.

The inner product as we initially learn it in Newtonian 3D space is the "dot product" which is typically defined by saying something like "multiply the corresponding components of the vectors together and add them all up."

In the "complex" spaces, $\mathbb{C}^n$ and L^2, we turn a ket into a bra using the same operation: complex conjugation. (In the discrete space we also transpose the column vector into a row vector, but this is really just a typographical nicety so that we can use matrix multiplication throughout.)

Take a look at the transition chart from last year for a summary of the relationship between these two spaces.

This suggests that with the appropriate definitions we can have an operation that turns a ket into a bra in the "real" spaces, such that all of our spaces can use $\sqrt{\langle v|v\rangle}$ for the metric. This does work out and the operation turns out to be *multiplication by the metric tensor.*

Upper and Lower Indices

Roughly speaking:
$v^\mu = |v\rangle$
$v_\mu = \langle v|$

There is some ambiguity in the use of this index notation. Sometimes v^μ is taken to mean the μth element of the vector v. But other times v^μ refers to the the entire vector, and the superscript μ is just letting you know that what *kind* of object it is (a ket as opposed to a bra). Which usage is intended *should* be clear from the context.

The Metric Tensor g

In relativistic physics, there is something called the *metric tensor.* It's a rank 2 tensor, or matrix. It's called the "metric tensor" because it embodies the metric of the space you're working in. I'm going to use the symbol g for the metric tensor. (Some people use different symbols in different spaces.)

For E^3 (regular 3D space), the metric tensor is: $g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

For M^4 (Minkowski spacetime): $g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Lowering and raising indices with g

In the complex spaces, $\mathbb{C}^n$ and L^2, we turn a ket into a bra using complex conjugation. In the real spaces, E^3 and M^4, we turn a ket into a bra by multiplying it with the metric tensor:

$$\langle v| = g|v\rangle$$

And the inner product can be written:

$$\langle v|w\rangle = g|v\rangle|w\rangle$$

If we want to retain the kind of conventions we've previously used, then it's understood that when we do this multiplication we also write the result as a row vector rather than a column vector. Once again this is not a necessity but it simply allows you to view the inner product as regular matrix multiplication.

For the most part people don't use Dirac notation in GR and we won't usually talk about bras and kets, but rather upper and lower indices and expressions using implicit summation. So we'll write the lowering operation like this:

$$v_\mu = g_{\mu\nu}v^\nu$$

And the inner product like this:

$$v_\mu w^\mu = g_{\mu\nu}v^\nu w^\mu$$

I said previously that sometimes indices are used to specify components of tensors and sometimes they simply indicate which kind of tensor it is (a ket or a bra, though those words aren't typically used). In the expressions above we are doing summation over components of the tensors. The indices specify individual components. However, if you interpret these expressions simply as a matrix multiplying a vector, it will work. You get the same results. But when we start using higher-ranked tensors you will definitely need to think about expressions like this as specifying component operations.

The metric tensor can always be inverted and its inverse is an upper-indexed tensor:

$$g^{-1} = g^{\mu\nu}, \quad g_{\mu\nu}g^{\mu\nu} = I$$

The upper-indexed metric tensor can be used to raise indices:

$$v^\nu = g^{\mu\nu}v_\nu$$

Appendix?

- Concoct an example in 3-D space. Vectors that do and don't work.
- "scalars" also must transform properly (example?)