The Ultraviolet Catastrophe and the Quantum Revolution - Plus Some Statistical Mechanics

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Physics Café

2025

Physics in the 19th Century

- Newtonian Classical Mechanics well established and successful.
- Young's slit experiments demonstrate the wave nature of light.
- Faraday's experiments with electromagnetism.
- Maxwell's equations unifying electromagnetism. Light is an electromagnetic wave with speed c = 3.00 * 10^8 m/s.
- Thermodynamics: steam engines and the Carnot cycle; Maxwell, Gibbs, and Boltzmann pioneer statistical thermodynamics: explaining thermodynamics from the perspective of the vas numbers of particles making up a system.

Physics circa 1900

- Many physicists believed they were close to a full understanding of the fundamentals of the physical universe.
- A few major outstanding problems to solve:
- Is matter really made of “indivisible” atoms?
- Reconciling Newtonian classical mechanics with Maxwell’s electromagnetism equations - relativity and the propagation of light.
- Photoelectric effect.
- Blackbody radiation and the ultraviolet catastrophe.

Summary of the Ultraviolet Catastrophe

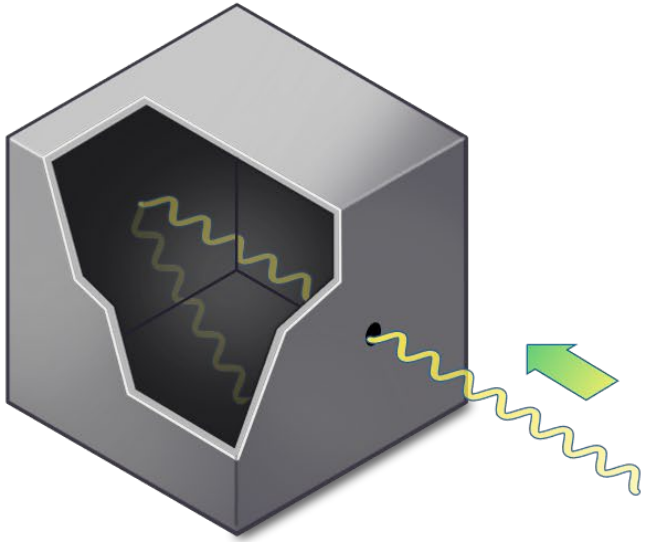
- Physicists attempted to derive, from first principles, formulas for how much EM energy objects emit at a given temperature, and its distribution across wavelengths/frequencies.
- Derivation using classical physics worked in the domain of long wavelengths/low frequencies – agreed with experiment.
- Classical physics broke down for short wavelengths/high frequencies.
- Distribution went to infinity – infinite energy.

Blackbody Radiation

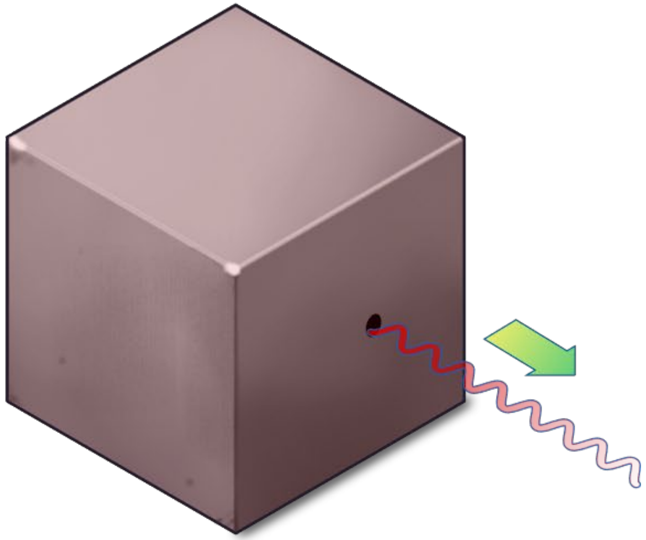
- Matter at temperatures greater than absolute zero emit electromagnetic waves (i.e. light).
- Light is emitted as spectrum of frequencies. Wavelength = c/frequency. Only a small fraction of the EM spectrum is visible light.
- A blackbody absorbs and emits all EM frequencies.
- Any frequency reflected by a body is not emitted by it: If two bodies are in thermal equilibrium, they must each transmit to other as much energy as they absorb from the other.

Blackbody Box

Blackbody Radiator

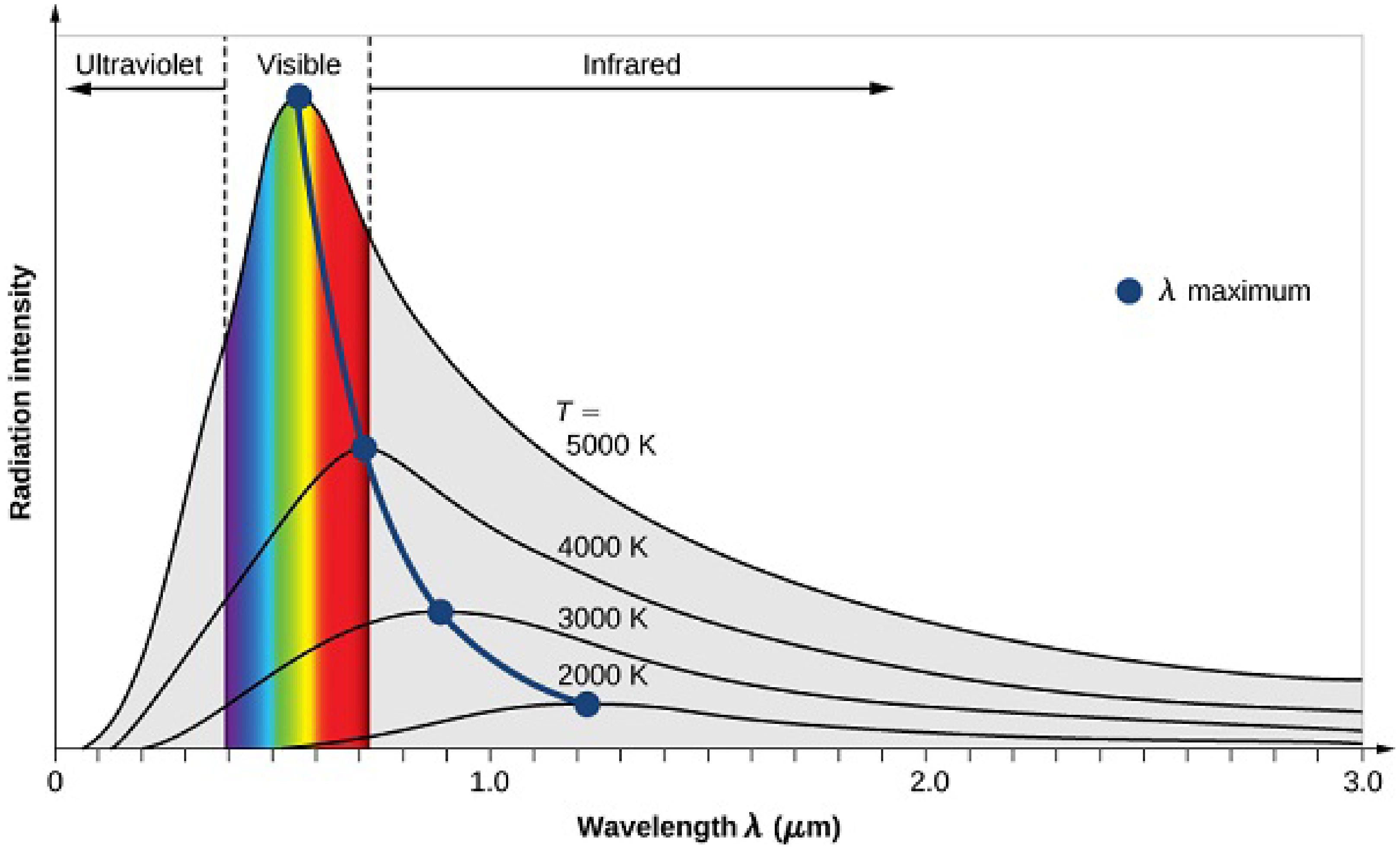


ll incident radiation is absorbed



Emitted radiation is only a function of radiator's temperature (T)

Blackbody Spectrum



Blackbody Experiments

- Experiments conducted by Wien and Lummer in 1895.
- Measure the spectrum intensity. Intensity is power (energy per unit time) per unit area.
- Intensity = c/4 * energy density (energy per unit volume in the box).
- The c/4 factor follows from a geometric derivation. The energy emitted is the amount, per unit time, reaching the cavity and travelling parallel to the normal vector of the cavity area. The energy within a hemispheric annulus at a constant radius, R, from the cavity, and travelling towards the cavity will reach it at the same time = R/c. Because of thermal equivalence principal mentioned before, the intensity is the same for all geometries.

Classical Derivation: Rayleigh-Jeans Law

- Count the wavelength states, the modalities, and add them weighted by their average energy.
- The walls of the box are conductive, so the electric field goes to zero.
- For a one dimensional situation, the length of the box must an integer multiple of half the wavelength.
- Wavelength (λ) = $2L/n$, where L is the length of the box and n is a positive integer. Frequency (f) = $c/\lambda = nc/(2L)$.

Traveling and Standing Waves

- Waves mathematically described using sines and cosines, or complex exponentials.
- The wave number (k) = $2\pi/\lambda$, and the angular frequency w = $2\pi f$.
- Right travelling wave equation: $\psi(x, t) = A * \exp[i(kx-wt)]$.
- Every wave traveling right matched by an equal and opposite wave travelling left: $\psi(x, t) = A * \exp[i(kx+wt)]$.
- Superposition principle: Total displacement equal to sum of the waves: $\psi(x, t) = A * (\exp[i(kx-wt)] + \exp[i(kx+wt)])$.
- $\mathrm{Re}[\psi(x, t)] = 2A * \cos(kx) * \cos(wt)$ – a standing wave.

Waves in 3 Dimensions

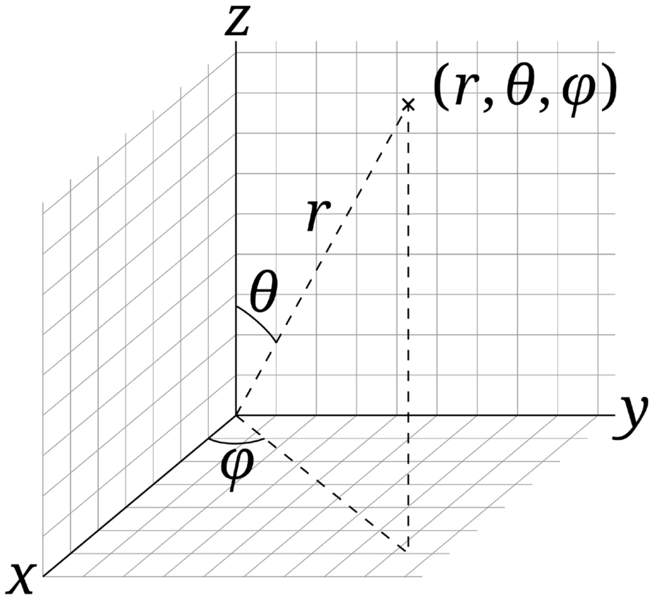
- Vector k – components in each of the three Cartesian directions, $\underset{k}{\rightarrow} = (k_x, k_y, k_z)$. $k_d = \frac{\pi n_d}{L}$, as the wave has to go to zero at the boundaries in all three dimensions. Assume a cubic box.
- Energy equals sum of modalities weighted by average energies:
- $E = 2 \sum_{n_z=1}^{\infty} \sum_{n_y=1}^{\infty} \sum_{n_x=1}^{\infty} < \varepsilon_{\{n_x, n_y, n_z\}} >$
- There is a factor 2 because each modality can come in two polarization states.

Changing Coordinates

- We can reasonably assume the energies are isotropic.
- For large n’s, the roughness of the sum of cubes is trivial.
- By the Pythagorean theorem: $\left|\underset{k}{\rightarrow}\right| = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{\pi}{L}\sqrt{n_x^2 + n_y^2 + n_z^2} \approx \frac{\pi n}{L}$
- Replace the discrete n_d’s with a single continuous n as a radial coordinate, and perform a volume integral over 1/8 of a sphere, as the sum was only over positive n’s.

Spherical Integral

- $E = 2\int_0^{\infty}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} < \varepsilon_n > \ n^2 \sin\theta \, d\varphi \, d\theta \, dn$
- $= 2\int_0^{\infty} < \varepsilon_n > n^2 dn \ \int_0^{\frac{\pi}{2}} \sin\theta \, d\theta \ \int_0^{\frac{\pi}{2}} d\varphi = \pi \ \int_0^{\infty} < \varepsilon_n > n^2 dn$
- It remains to figure out $< \varepsilon_n >$.



Statistical Mechanics Intro

- Understanding thermodynamics from statistics of enormous numbers of particles.
- Pioneered by: Ludwig Eduard Boltzmann, James Clerk Maxell, and Josiah Willard Gibbs in the latter half of the 19th century.
- For every observable macrostate, there are one or more microstates – specific arrangement of particles and energy.
- All possible microstates are equally probable – therefore the probabilities of macrostates depend on the number of corresponding microstates (Ω).

Introductory Example

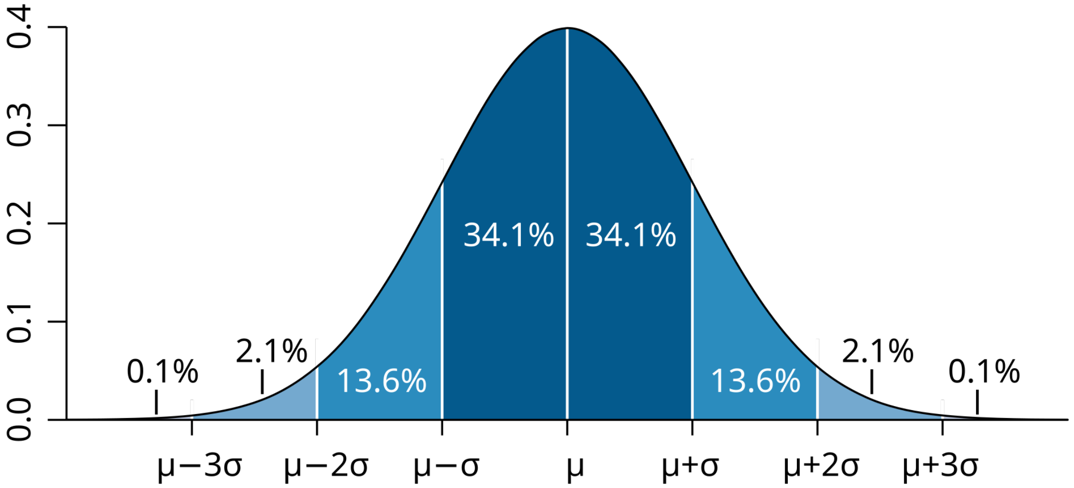
- Consider N identical, non-interacting gas molecules in a sealed chamber. How many are on the left side (N_L), and how many are on the right side (N_R), assuming the volumes of both sides are the same?
- We can use the binomial distribution.
- There are a total of 2^N possible microstates, as there are 2 possibilities for each molecule.
- For any macrostate, the total number of molecules on each side, the number of microstates is $\binom{N}{N_L} = \binom{N}{N_R} = \frac{N!}{N_L!N_R!}$

Introductory Example

- Probability(N_L, N_R) = $\frac{N!}{N_L! N_R!} * \frac{1}{2^N}$
- The probabilities of all molecules being on the left, or all on the right side, is the same as any other microstate, but unlikely, because macrostates with an even distribution, or close to it, have many more corresponding microstates.
- For N = 20, the chance of all molecules being on one specified side is less than one a million, compared to a 17.6% chance of having 10 on each side, and a 50% chance of there being 9-11 on one specified side.

Introductory Example

- For large N, the distribution of the proportion follows a Gaussian "bell curve", with a mean of ½ and a standard deviation $^{1}/_{(2\sqrt{N})}$.
- Approximately 68% probability of being within 1 standard deviation of the mean, 95% probability of being within 2 standard deviations of the mean, and almost always within 3 standard deviations of the mean.
- As N increases, the distribution becomes more narrow.



Maximum Entropy Principle

- For macroscopic objects, N is typically on the order of Avogadro's number: 6.022 * 10^23.
- Only macrostates at, or very close to, the maximum multiplicity have any appreciable probability of occurring.
- Entropy was defined by Boltzmann as $S = k_B \, LN[\Omega]$, where Ω is the multiplicity and k_B is the Boltzmann constant ≈
- 1.381 *10^(-23) J/K = 8.617 * 10^(-5) eV/K.
- Macroscopic systems in thermal equilibrium are always in a state of maximum entropy.

Maximum Entropy Principle

- Logarithm converts very large numbers (i.e. 10^(10^23)) into merely large numbers (i.e. 10^23).
- Entropy is additive, whereas multiplicity is multiplicative – doubling the system doubles the entropy, whereas multiplicity is squared.
- The Boltzmann constant relates entropy to energy and temperature.
- $\frac{\alpha S}{\alpha U} = \frac{1}{T}$, $\mathrm{d}S = \frac{dQ}{T}$, where U is internal energy, Q is heat input, and T is temperature, with zero being absolute zero (i.e. Kelvin scale).
- Second law of thermodynamics – entropy of an isolated system increase with time.
- Heat flows from hot to cold because the hot object loses less entropy than the cold object gains.

Ludwig Eduard Boltzmann

Ludwig Eduard Boltzmann

- Austrian mathematician and physicist (1844-1906).
- One of the founders of statistical mechanics.
- Studied under Stefan, who introduced him to Maxwell’s physics.
- Professor at the universities of Graz and Vienna.
- Supported Henriette von Aigentler’s studies of mathematics and physics when she faced resistance for being a women – married her – had three daughters and a son.
- Mentored Paul Ehrenfet and Elise “Lise” Meitner – female physicist who later co-discovered nuclear fission.

Ludwig Eduard Boltzmann

- Popular lecturer on physics and philosophy, lectured at Berkley in 1905, described his philosophy as materialism.
- Conflict with colleagues, particularly Ernst Mach, and many German philosophers, who rejected the existence of atoms.
- Mental health problems: depression, possibly bipolar.
- Committed suicide by hanging in 1906 (aged 62), believing his life’s work had been for naught. Completely vindicated a few years later.
- His entropy equation: $S = k * \log W$ written on his grave.

Boltzmann Statistics

- Imagine a system, say a single particle, in thermal equilibrium with a much larger reservoir. Only energy is exchangeable.
- Let the particle have an energy of ε_0 when in state s_0, and an energy of ε_1 when in state s_1.
- Prob(s_1)/Prob(s_0) = exp[-(ε_1 - ε_0)/(k_BT)].
- Prob(s_1)/Prob(s_0) = Ω_{R_1} / Ω_{R_0} = exp[$(S_{R_1}$ - $S_{R_0})$/ k_B]
- $\approx \exp\left[\frac{1}{k_B} * \frac{dS_R}{dE_R} * -(\varepsilon_1 - \varepsilon_0)\right] = \exp\left[\frac{1}{k_B} * \frac{1}{T} * -(\varepsilon_1 - \varepsilon_0)\right]$

Boltzmann Statistics

- To convert to absolute probabilities, we need to normalize:
- $Prob(s) = \exp[-\varepsilon_s/k_BT]/Z$, $\mathrm{Z} = \sum_s \exp[-\varepsilon_s/k_BT]$, since the total probability must sum to one.
- Adding a constant to all energies has no effect on the probabilities, as it would cancel out.
- Comports with intuition – nature tends to run downhill, seeking lower energy states – so high energy states should be relatively rare.
- Sum is over states. Degeneracy, states with same energy, need to be taken into account. Degeneracy often varies with energy.

Boltzmann Statistics – Maximization Derivation

- Stirling's approximation: for large N, N! $\approx N^N e^{-N} \sqrt{2\pi N}$, and $LN[N!] \approx N\ LN[N] - N$.
- Entropy can be expressed as $S = -k_B * \sum_s prob_s * LN[prob_s]$.
- Maximize entropy in terms of the state probabilities, with constraints of the probabilities adding up to one, and expected energy being constant:
- $\sum_s prob_s = 1$ and $\sum_s \varepsilon_s\, prob_s = \bar{\varepsilon}$.
- Apply the method of Lagrange multipliers for constrained optimization.
- Maximization probabilities are the Boltzmann factors.

Partition Function

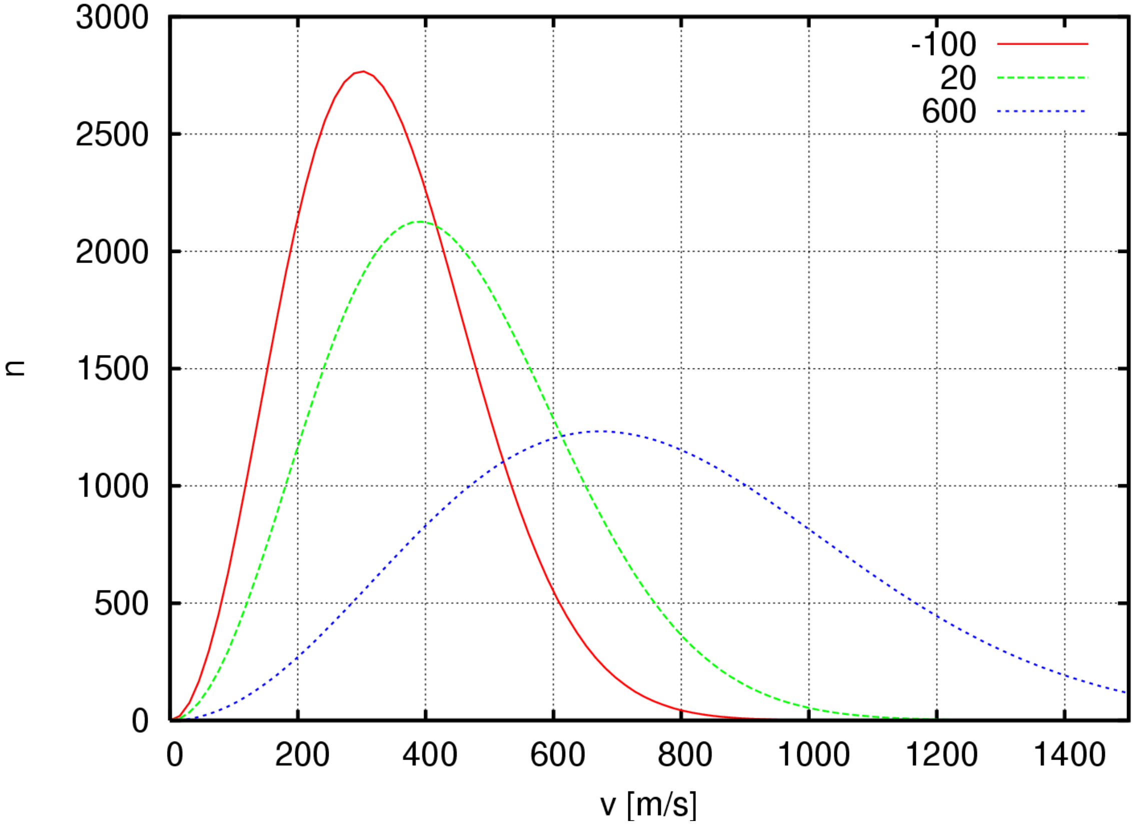
- $\mathrm{Z} = \sum_s \exp[-\varepsilon_s / k_B T]$, the sum of the Boltzmann factors, is called the partition function. Depends on temperature.
- For convenience, rewrite as $\mathrm{Z} = \sum_s \exp[-\beta \varepsilon_s]$, where
- $\beta = 1/(k_B T)$.
- Derivatives of the partition function with respect to β can be used to calculate the average and other moments for energy.
- $\bar{\varepsilon} = -\frac{\alpha LN[Z]}{\alpha\beta} = -\frac{1}{Z}\frac{\alpha Z}{\alpha\beta} = \sum_s \varepsilon_s * \exp[-\beta\varepsilon_s]/Z = \sum_s \varepsilon_s * prob_s$

Combining Partition Functions

- N distinguishable particles: $Z_{total} = \prod_{i=1}^{N} Z_i$
- N indistinguishable particles: $Z_{total} \approx \prod_{i=1}^{N} Z_i / N!$
- Factorial corrects for permutations, but the formula is approximate because it ignores the possibility of multiple particles being in the same state.
- Valid approximation for most classical situations which are not “too dense” (ex. ideal gas). The number of possible states, which are on the Planck scale, is much larger than the number of particles, so the probability of multiple particles in the same state is negligible. Different for quantum systems.

Maxwell-Boltzmann Distribution

- Distribution of speed of particles gas.
- Example graph is for one million oxygen molecules at -100, 20 and 600 degrees Celsius.



Maxwell-Boltzmann Distribution

- $Prob(v_1 < v < \ v_2) = \frac{1}{C}\int_{v_1}^{v_2} g(v) * \exp[-\beta E_{tr}(v)]\, dv$
- $E_{tr}(v) = \ {}^{mv^2}\!/_{2}$ - translation kinetic energy – depends on the square magnitude of velocity.
- $\beta = \ 1/(k_B T)$
- $g(v) = 4\pi v^2$ is the degeneracy factor – surface area of a sphere since a velocity vector – with a fixed magnitude - can point in any 3 dimensional direction.
- $C = \int_0^{\infty} g(v) * \exp[-\beta E_{tr}(v)]\, dv$ is the normalizing factor.

Integration Method

- $I(\alpha, p) = \int_0^{\infty} x^{2p} * e^{-\alpha x^2} dx$ - depends on α and p, α > 0
- Consider $I(\alpha, 0) = \int_0^{\infty} e^{-\alpha x^2} dx.$
- $I(\alpha, 0) = \frac{1}{2} * \int_{-\infty}^{\infty} e^{-\alpha x^2} dx = \frac{1}{2}\sqrt{\left[\int_{-\infty}^{\infty} e^{-\alpha x^2} dx\right]^2} = \frac{1}{2}\sqrt{\left[\int_{-\infty}^{\infty} e^{-\alpha x^2} dx\right] * \left[\int_{-\infty}^{\infty} e^{-\alpha y^2} dy\right]} = \frac{1}{2}\sqrt{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\alpha(x^2+y^2)} dxdy}$
- Change to polar coordinates: $I(\alpha, 0) = \frac{1}{2}\sqrt{\int_0^{2\pi}\int_0^{\infty} e^{-\alpha r^2} r dr d\theta}$

Integration Method

- $I(\alpha, 0) = \frac{1}{2}\sqrt{2\pi \ * 1/(2\alpha)} = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$
- For integer p > 0, take derivatives with respect to α.
- $I(\alpha, p) = \int_0^\infty x^{2p} * e^{-\alpha x^2} dx = \int_0^\infty (-1)^p \frac{d^p(e^{-\alpha x^2})}{d\alpha^p}$
- $= (-1)^p \frac{d^p}{d\alpha^p} \int_0^\infty e^{-\alpha x^2} dx = (-1)^p \frac{d^p I(\alpha,0)}{d\alpha^p}$
- $= \frac{\sqrt{\pi}}{2} * \frac{\prod_{j=1}^{p}(2j-1)}{2^p} * \alpha^{(-(1+2p)/2)}, p = \{1, 2, \ldots\}$

Maxwell-Boltzmann Distribution Continued

- $C = \int_0^\infty g(v) * \exp[-\beta E_{tr}(v)]\, dv =$
- $\int_0^\infty (4\pi v^2) * \exp\left[-mv^2/(2k_B T)\right] dv = 4\pi * \frac{\sqrt{\pi}}{2} * \frac{1}{2} * \left(\frac{2k_B T}{m}\right)^{3/2}$
- $= \left(\frac{2\pi k_B T}{m}\right)^{3/2}$
- $Prob(v_1 < v < v_2) = \left(\frac{m}{(2\pi k_B T)}\right)^{3/2} \int_{v_1}^{v_2} (4\pi v^2) * \exp\left[-mv^2/(2k_B T)\right] dv$

Average Energy of Monoatomic Gas

- Monoatomic gas (ex. Helium) – only energy is translational
- $< KE > = \left(\frac{m}{(2\pi k_B T)}\right)^{3/2} \int_0^\infty \left(\frac{mv^2}{2}\right) * \left(4\pi v^2\right) * \exp\left[-mv^2/(2k_B T)\right] dv$
- $= \left(\frac{m}{(2\pi k_B T)}\right)^{3/2} * \frac{m}{2} * 4\pi * \frac{\sqrt{\pi}}{2} * \frac{3}{4} * \left(\frac{2k_B T}{m}\right)^{5/2} = 3k_B T/2$
- Root mean square velocity: $\frac{mv_{rms}^2}{2} = 3k_B T/2\,, v_{rms} = \sqrt{3k_B T/m}$

Equipartition Theorem

- Energy is equally distributed among degrees of freedom.
- Break down the integral into Cartesian coordinates:
- $< KE > = \frac{1}{C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{m(v_x^2+v_y^2+v_z^2)}{2}\right) * \exp[-m(v_x^2+v_y^2+v_z^2)/(2k_BT)]\,dv_x dv_y dv_z$
- $C = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp[-m(v_x^2+v_y^2+v_z^2)/(2k_BT)]\,dv_x dv_y dv_z$
- Consider the x component, the others follow by symmetry:
- $< KE_x > = \frac{1}{C}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\frac{mv_x^2}{2}\right) * \exp[-m(v_x^2+v_y^2+v_z^2)/(2k_BT)]\,dv_x dv_y dv_z$

Equipartition Theorem

- Rearrange the integrals:
- $< KE_x >= \frac{1}{C} * \frac{m}{2} * 2\int_0^{\infty} v_x^2 * \exp\left[\frac{-mv_x^2}{2k_BT}\right] dv_x \int_{-\infty}^{\infty} \exp\left[\frac{-mv_y^2}{2k_BT}\right] dv_y \int_{-\infty}^{\infty} \exp\left[\frac{-mv_z^2}{2k_BT}\right] dv_z$
- And $C = 2\int_0^{\infty} \exp\left[\frac{-mv_x^2}{2k_BT}\right] dv_x \int_{-\infty}^{\infty} \exp\left[\frac{-mv_y^2}{2k_BT}\right] dv_y \int_{-\infty}^{\infty} \exp\left[\frac{-mv_z^2}{2k_BT}\right] dv_z$
- Integrals over y and z cancel out.

Equipartition Theorem

- $< KE_x > = \frac{m}{2} * \int_0^\infty v_x^2 * \exp\left[\frac{-mv_x^2}{2k_BT}\right] dv_x \Big/ \int_0^\infty \exp\left[\frac{-mv_x^2}{2k_BT}\right] dv_x$
- $= \frac{m}{2} * \left(\frac{\sqrt{\pi}}{2} * \frac{1}{2} * \left(\frac{2k_BT}{m}\right)^{3/2}\right) \Big/ \left(\frac{\sqrt{\pi}}{2} * \left(\frac{2k_BT}{m}\right)^{1/2}\right) = \frac{m}{4} * \frac{2k_BT}{m} = \frac{1}{2}k_BT$, same for $< KE_y >$ and $< KE_z >$.
- Each degree of freedom has an average energy of $\frac{1}{2}k_BT$.
- Applies more generally to where $E = \sum_j c_j q_j^2$, where the q’s are canonical coordinates (position, momentum, angular momentum, etc..) and the c’s are constant scalers.

Applications of the Equipartition Theorem

- Diatomic gases (ex. Nitrogen, Oxygen) – three translation degrees of freedom and two rotational degrees of freedom.
- Note: If molecules were truly mechanical there would be three rotational degrees of freedom, but one is omitted because of quantum mechanical reasons.
- $< KE > = < KE_{translational} > \; + < KE_{rotational} > =$
- $3\left(\frac{1}{2}k_B T\right) + 2\left(\frac{1}{2}k_B T\right) = \frac{5}{2}k_B T$

Applications of the Equipartition Theorem

- Harmonic oscillator – 2 degrees of freedom
- $E = \frac{mv^2}{2} + \frac{kx^2}{2}, < E >= 2\left(\frac{1}{2}k_B T\right) = k_B T$
- An electro-magnetic wave can be analogized to a harmonic oscillator.
- Not simple because the electric and magnetic field are not independent.

A Sketch of the EM Wave Hamiltonian

- Electromagnetic energy density: $\frac{\varepsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2$
- Where E and B are the electric and magnetic fields respectively, and ε_0 and μ_0 are the electric and magnetic permittivity constants respectively.
- In the absence of electric charges or currents, Maxwell's electromagnetism equations can be written as:
- $\underset{B}{\rightarrow} = \nabla X \underset{A}{\rightarrow}$, and $\underset{E}{\rightarrow} = \frac{-1}{c}\frac{\alpha \underset{A}{\rightarrow}}{\alpha t}$, where $\underset{A}{\rightarrow}$ is the vector potential, and c is the speed of light $= 1/\sqrt{\varepsilon_0 \mu_0}$

A Sketch of the EM Wave Hamiltonian

- and $\underset{A}{\rightarrow}$ satisfies the standard wave equation:
- $\nabla^2 \underset{A}{\rightarrow} - \frac{1}{c^2}\frac{\alpha^2 \underset{A}{\rightarrow}}{\alpha t^2} = 0$
- $\underset{A}{\rightarrow}$ can be decomposed into a Fourier series, with coefficients of the form: $Q_\lambda + iP_\lambda$, where λ is the wavelength, the Hamiltonian for each wavelength is of the form: $Q_\lambda^2 + P_\lambda^2$, and the total Hamiltonian involves a sum over the wavelengths.
- Q and P are a canonical Hamiltonian parameter pair, analogous to position and momentum.

A Sketch of the EM Wave Hamiltonian

- In the Hamiltonian formulation:
- $\frac{dQ_\lambda}{dt} = \frac{\alpha H}{\alpha P_\lambda}$, and $\frac{dP_\lambda}{dt} = -\frac{\alpha H}{\alpha Q_\lambda}$, where H is the Hamiltonian.
- Satisfies Maxwell's equations and the wave equation.
- The form of the EM wave Hamiltonian for a modality is identical to a harmonic oscillator: one position term squared and one corresponding momentum term squared.
- Can apply the equipartition theorem result for a harmonic oscillator to an EM wave modality. Average energy $= \ k_B T$.

Blackbody Spectrum – Classical Derivation

- Recall: $E = \pi \int_0^\infty < \varepsilon_n > n^2 dn.$
- Substitute $< \varepsilon_n > = k_B T$ from the equipartition theorem and harmonic oscillator analogy.
- $E = \int_0^\infty \pi k_B T \ * \ n^2 dn$
- Change variables to frequency or wavelength for a more physical meaning.

Blackbody Spectrum – Classical Derivation

- Frequency (f) = nc/(2L), df = c/(2L) * dn, n = 2Lf/c, dn = 2L/c * df
- $E = L^3 \int_0^\infty 8\pi k_B T / c^3 * \, f^2 df$
- Wavelength (λ) = 2L/n, dλ $= \; -2L/n^2 \, dn$, n = 2L/λ,
- $dn = -\,(2L/\lambda)^2/2L * d\lambda = -2L \; * \; d\lambda/\lambda^2$
- $E = L^3 \int_\infty^0 8\pi k_B T * \; 1/\lambda^4 \; * (-d\lambda) = \; L^3 \int_0^\infty 8\pi k_B T * \; 1/\lambda^4 \; * d\lambda$

Blackbody Spectrum – Classical Derivation

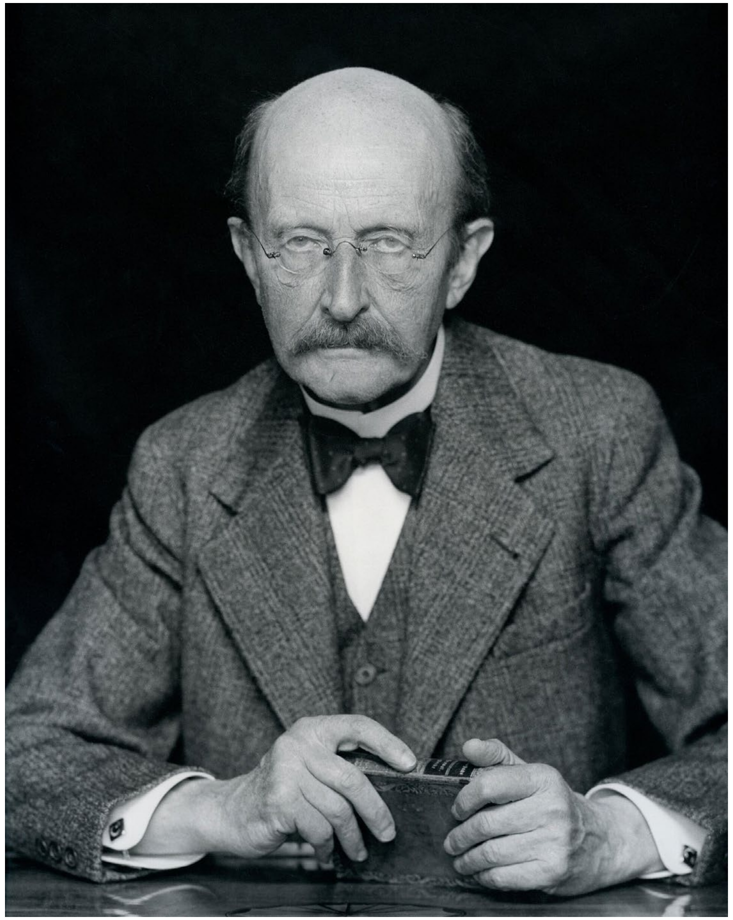
- Note: L^3 is the volume, so factoring it out gives us the energy density.
- Recall Intensity = c/4 * energy density.
- $I = \int_0^\infty 2\pi k_B T/c^2 * \ f^2 df \ = \int_0^\infty 2\pi c k_B T * \ 1/\lambda^4 \ * d\lambda$
- Agrees well with experiment for long wavelengths/low frequencies.

Ultraviolet Catastrophe

- For short wavelengths/high frequencies – ultraviolet, beyond visible light - the formula increases without bound.
- Integral is not convergent.
- Real objects are not blasting out an infinite of energy in the form of gamma rays.
- Classical physics inadequate for short wavelengths/high frequencies.

Enter Max Planck

- German theocratical physicist (1858- 1947)
- Reluctant revolutionary.
- The ultraviolet catastrophe was such an important and vexing problem he decided it was necessary to make a fundamental change in the understanding of physics.
- Quantize energy – energy comes in discrete packets (quanta).
- Won the 1918 Nobel Prize in Physics for quanta.



Quantization of Energy

- For a given modality (n), let all energies be an integer multiple of some underlying energy.
- $\epsilon_{n,j} = j \; * \epsilon_{n,1}, j = \{0,1,2,\dots\}$
- Changes the average energy calculation – replace integrals with sigmas.
- $< \varepsilon_n > = \left(\sum_{j=0}^{\infty} \varepsilon_{n,j} \; * \exp[-\beta \varepsilon_{n,j}]\right) / \left(\sum_{j=0}^{\infty} \exp[-\beta \varepsilon_{n,j}]\right)$
- $= \left(\sum_{j=0}^{\infty} \left(j\epsilon_{n,1}\right) * \exp[-\beta\left(j\epsilon_{n,1}\right)]\right) / \left(\sum_{j=0}^{\infty} \exp[-\beta\left(j\epsilon_{n,1}\right)]\right)$

Quantization of Energy

- For the denominator: $\sum_{j=0}^{\infty} \exp[-\beta(j\epsilon_{n,1})] = \sum_{j=0}^{\infty}(\exp[-\beta\varepsilon_{n,1}])^{j}$
- $= 1/(1 - \exp[-\beta\varepsilon_{n,1}])$, by the formula for a geometric sum.
- For the numerator: $\sum_{j=0}^{\infty}(j\epsilon_{n,1}) * \exp[-\beta(j\epsilon_{n,1})]$
- $= \sum_{j=0}^{\infty} -d(\exp[-\beta(j\epsilon_{n,1})])/d\beta = -d\left(\sum_{j=0}^{\infty} \exp[-\beta(j\epsilon_{n,1})]\right)/d\beta$
- $= -d(1/(1 - \exp[-\beta\varepsilon_{n,1}]))/d\beta$
- $= 1/(1 - \exp[-\beta\varepsilon_{n,1}])^{2} * \varepsilon_{n,1} * \exp[-\beta\varepsilon_{n,1}]$

Quantization of Energy

- $< \varepsilon_n > = \left(\varepsilon_{n,1} * \exp[-\beta\varepsilon_{n,1}]/\left(1 - \exp[-\beta\varepsilon_{n,1}]\right)^2]\right)/\left(1/\left(1 - \exp[-\beta\varepsilon_{n,1}]\right)\right)$
- $= \varepsilon_{n,1} * \exp[-\beta\varepsilon_{n,1}]/\left(1 - \exp[-\beta\varepsilon_{n,1}]\right) = \varepsilon_{n,1}/\left(\exp\left[\frac{\varepsilon_{n,1}}{k_B T}\right] - 1\right)$
- If $\varepsilon_{n,1}$ is small: $< \varepsilon_n > \approx \varepsilon_{n,1}/\left(1 + \frac{\varepsilon_{n,1}}{k_B T} - 1\right) = k_B T$, classical limit.
- If $\varepsilon_{n,1}$ is large: $< \varepsilon_n > \rightarrow 0 \; as \; \varepsilon_{n,1} \rightarrow \infty$, as the exponential grows faster than the linear term.

Quantization of Energy

- For the blackbody spectrum, we want: $< \varepsilon_n > \approx k_B T$ for long wavelengths/low frequencies, and $< \varepsilon_n > \to 0$ for short wavelengths/high frequencies.
- Set $\varepsilon_{n,1} = hf = hc/\lambda$.
- h is called the Plank constant ≈ 6.626 * 10^(-34) J*s
- = 4.136 * 10^(-15) eV/Hz.
- Quanta of light called photons.
- $< \varepsilon_n > = hf / \left(\exp\left[\frac{hf}{k_B T} \right] - 1 \right) = \frac{hc}{\lambda} * 1 / \left(\exp\left[\frac{1}{k_B T} * \frac{hc}{\lambda} \right] - 1 \right)$.

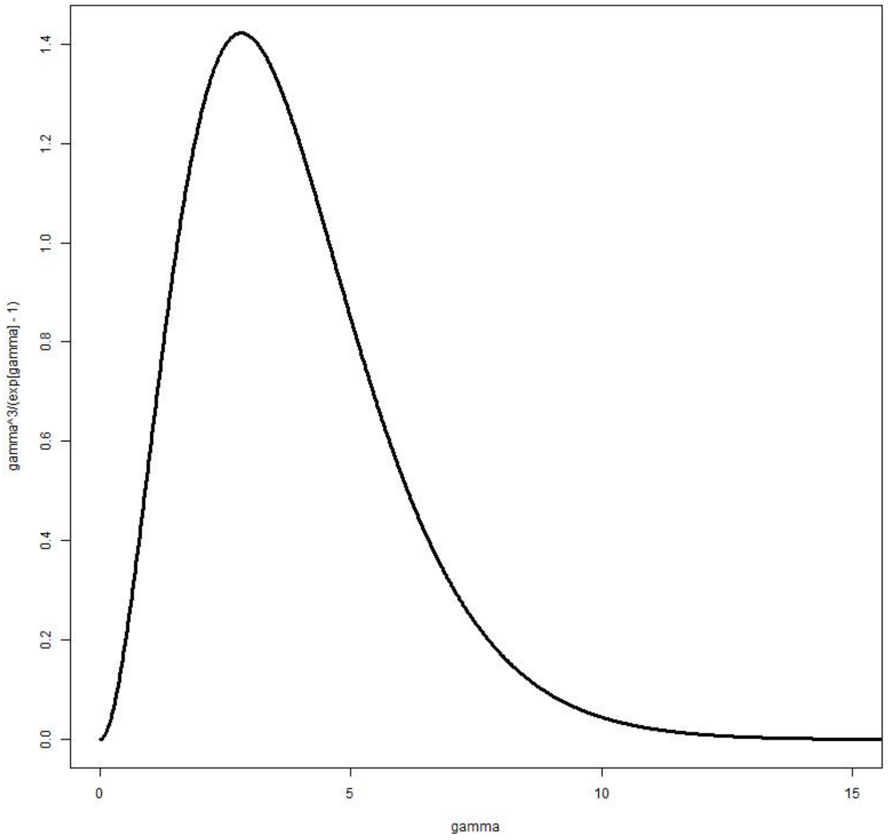
Plank's Blackbody Spectrum

- $I = \int_0^\infty 2\pi/c^2 * hf/\left(\exp\left[\frac{hf}{k_B T}\right] - 1\right) * f^2 df =$
  $\int_0^\infty 2\pi h/c^2 * f^3/\left(\exp\left[\frac{hf}{k_B T}\right] - 1\right) * \ df$
  $= \int_0^\infty 2\pi c * \frac{hc}{\lambda} \ * 1/\left(\exp\left[\frac{1}{k_B T} \ * \ \frac{hc}{\lambda}\right] - 1\right) * 1/\lambda^4 \ * d\lambda$
  $$= \int_0^\infty 2\pi hc^2 * 1/\left(\exp\left[\frac{1}{k_B T} \ * \ \frac{hc}{\lambda}\right] - 1\right) * 1/\lambda^5 \ * d\lambda$$

Plank’s Blackbody Spectrum

- In terms of quantas: $\varepsilon = hf, \mathrm{d}\epsilon = h \ast df, f = \frac{\varepsilon}{h}, df = \frac{d\epsilon}{h}$
- $I = \int_0^\infty 2\pi/\left(h^3c^2\right) \ast \varepsilon^3/\left(\exp\left[\frac{\varepsilon}{k_BT}\right] - 1\right) \ast\ d\varepsilon$
- No longer explodes for short wavelengths/high frequencies.
- To evaluate the integral, convert to a dimensionless quantity:
- $\gamma = \frac{\varepsilon}{k_BT}, d\gamma = \frac{d\varepsilon}{k_BT}, \varepsilon = k_BT\gamma, d\varepsilon = k_BT \ast d\gamma$
- $I = 2\pi(k_BT)^4/\left(h^3c^2\right) \ast \int_0^\infty \gamma^3/(\exp[\gamma] - 1) \ast\ d\gamma$

Plank’s Blackbody Spectrum



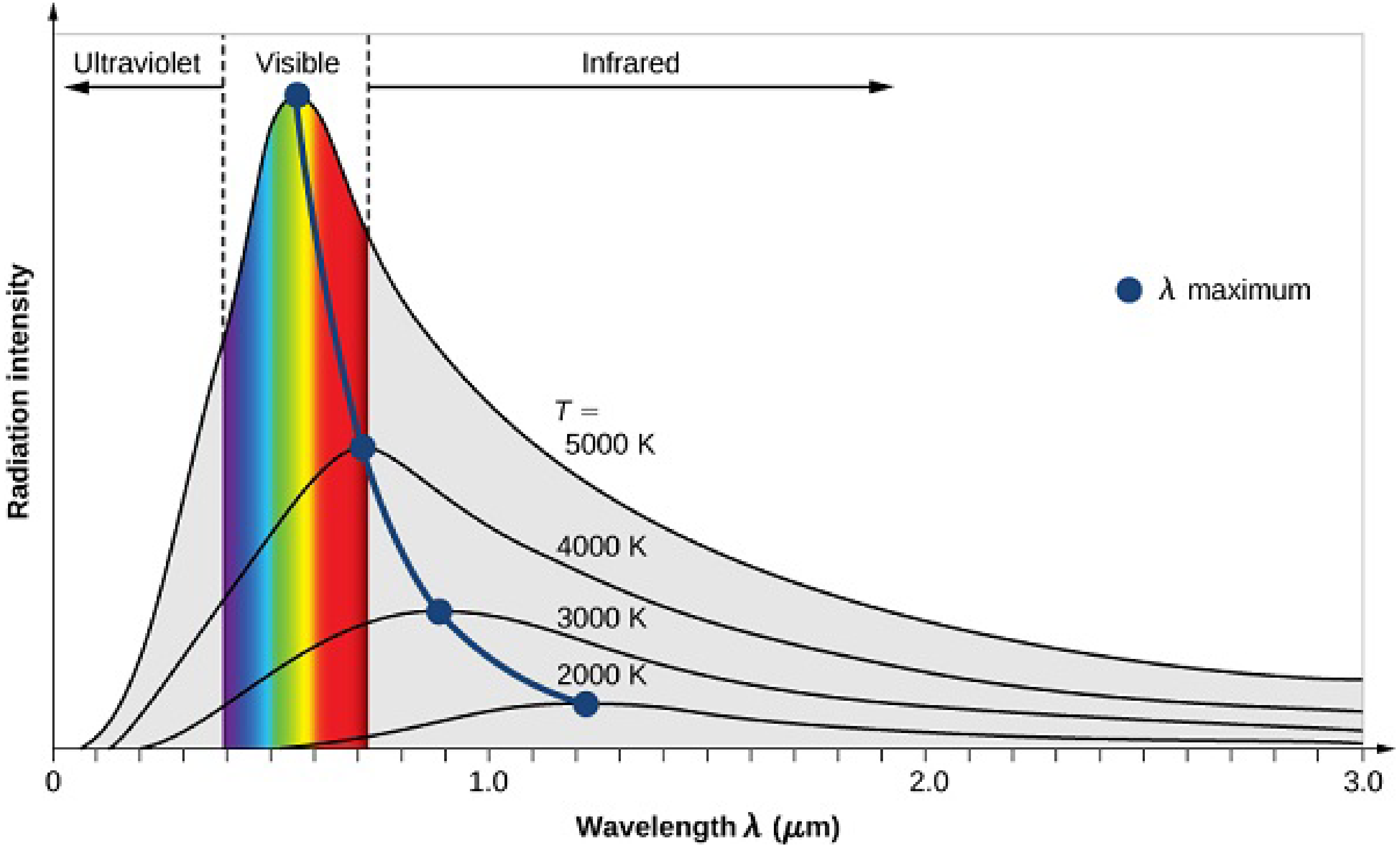
Plank’s Blackbody Spectrum

- $\int_0^\infty \gamma^p/(\exp[\gamma] \pm 1) * d\gamma\ , p = \{0, 1, 2, \ldots\}$ difficult to evaluate for p > 0. There is a method for analytically solving it for an odd p. Even p’s require numeric integration.
- For p = 3, $\int_0^\infty \gamma^3/(\exp[\gamma] - 1) * d\gamma = \frac{\pi^4}{15}$.
- $I = 2\pi^5 k_B^4/\left(15h^3c^2\right) * T^4 = \sigma T^4$, where σ is the Stefan-Boltzmann constant.
- $\sigma \approx 5.670 * 10^{-8}\ \frac{W}{m^2K^4}$, radiation output proportional to the fourth power of temperature.

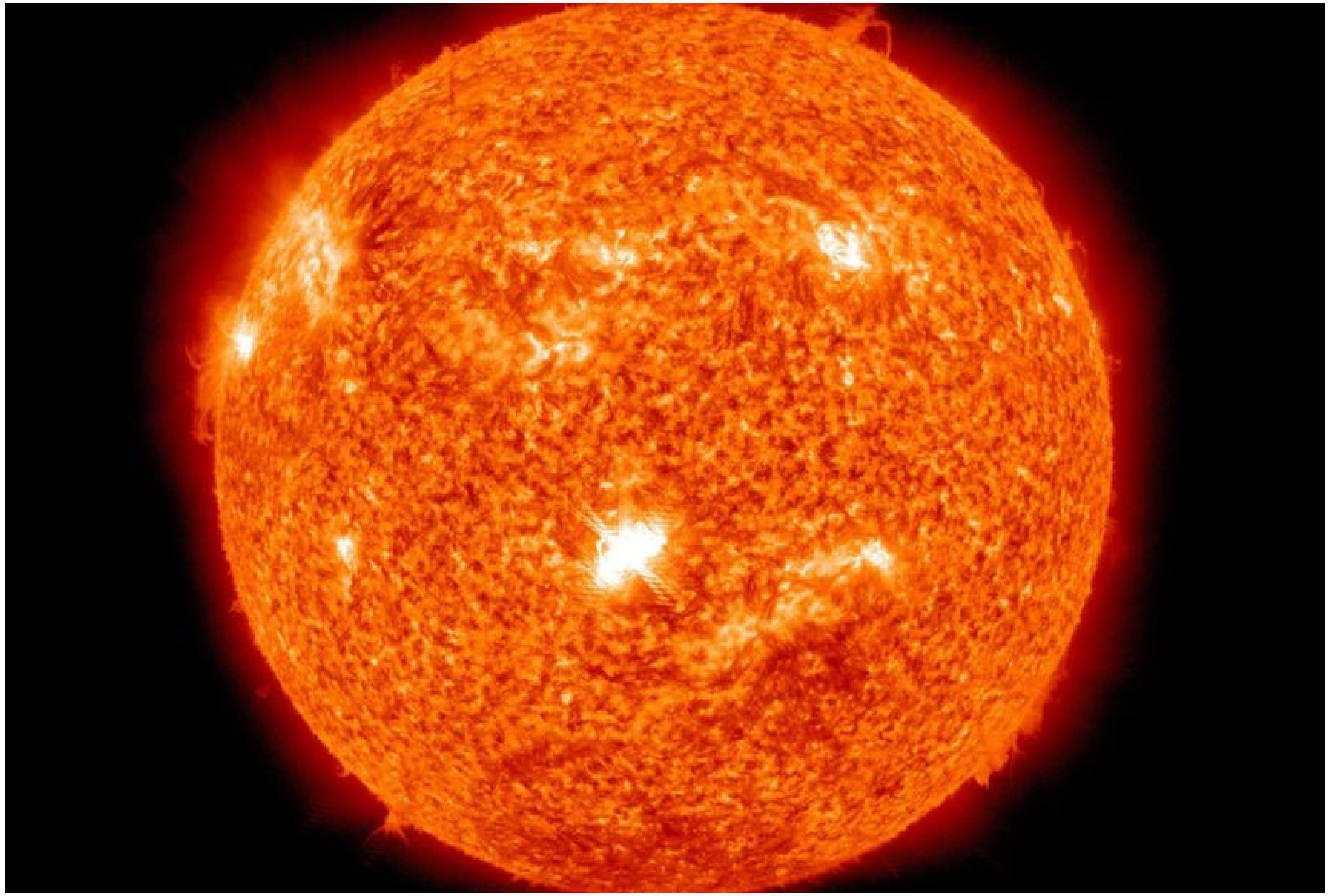
Plank’s Blackbody Spectrum

- Excellent match to experimental results.
- Spectrum peaks for $\gamma \approx 2.82$.
- The peak for energy is at $\varepsilon \approx 2.82 k_B T$.
- The peak for frequency is at $f \approx 2.82 k_B T/h$.
- The peak for wavelength is at $\lambda \approx b/T$, Wien’s law.
- $b \approx 2.898 * 10^{-3}\, mK$, called Wien’s displacement constant.
- Higher temperatures mean more power per area, higher peak energies and frequencies, and shorter peak wavelengths.

Plank's Blackbody Spectrum With Wien's Law



Example: Solar Radiation



Example: Solar Radiation

- The sun is approximately a spherical blackbody.
- Surface Temperature = 5772 K.
- Radius = 6.96 * 10^8 m (109 times Earth's (6.37 * 10^6 m)).
- Surface Area = 4π * (radius)^2 = 6.09 * 10^18 m^2 (12,000 times Earth's).
- Earth-Sun Distance = 1.50 *10^11 m (1 AU by definition).
- Luminosity = $\sigma T^4 \ * (surface\ area) = \left(5.670 \ * 10^{-8} \ \frac{W}{m^2 K^4}\right) \ * \ (5772\ K)^4 \ * \ \left(6.09 * 10^{18}\ m^2\right) = 3.83 \ * 10^{26} W$

Example: Solar Radiation

- Intensity at Earth orbit = Luminosity/(4π * (Earth-Sun distance)^2) = $\left(3.83 \ * 10^{26} W\right)/\left(4\pi \ * \ \left(1.50 \ * 10^{11} \ m\right)^2\right) = \ 1.36 \ * \ 10^3 \ \frac{W}{m^2}$.
- Solar radiation received by the Earth = Intensity at Earth * cross sectional area $= \left(1.36 \ * \ 10^3 \ \frac{W}{m^2}\right) \ * \ \left(\pi \left(6.37 \ * 10^6 \ m\right)^2\right) = 1.73 \ * \ 10^{17} \ W$
- Peak frequency =$2.82 k_B T/h = 2.82 \ * \left(1.381 * 10^{-23} \ \frac{J}{K}\right) * (5772 \text{ K})/\left(6.626 * 10^{-34} \text{ J}*\text{s}\right) = 3.39 \ * 10^{14} \ Hz$

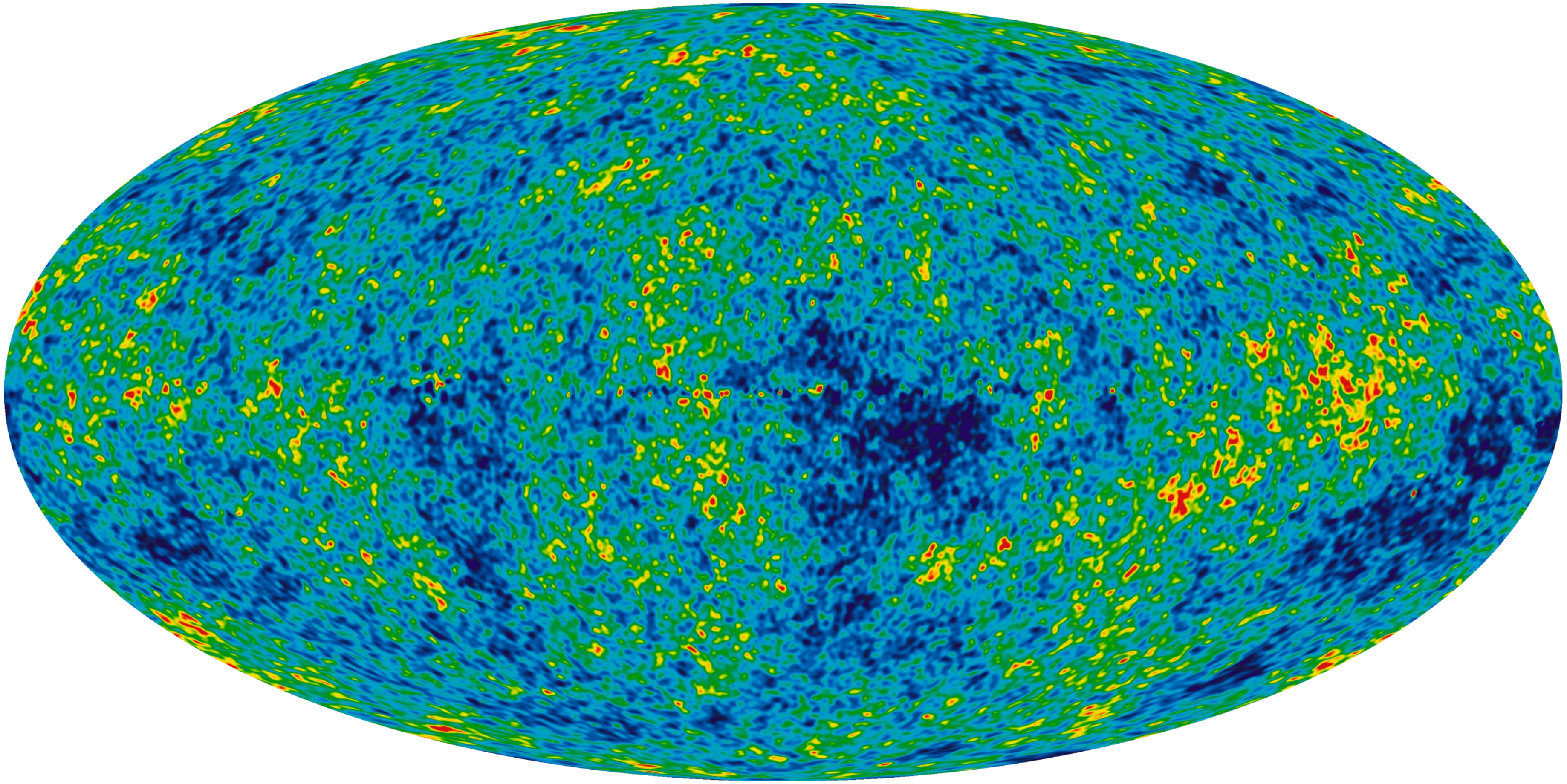
Example: Solar Radiation

- Peak Energy for Photons = $2.82k_BT = 2.82 * \left(8.617 * 10^{-5} \ \frac{eV}{K}\right) * (5772\ K) = 1.40\ eV.$
- Peak Wavelength = $b/T = \ 2.898 \ * \ 10^{-3}\ mK/(5772\ K) = 5.02 * 10^{-7}\ m = 502\ nm$ $(598 \text{ tera Hz}$, 2.47 eV for photons) – green light, close to blue.

Example: Solar Radiation

- Sunlight in space approximately 50% infrared, 40% visible (750-400 nm), and 10% ultraviolet.
- Light at the Earth’s surface affected by axial tilt, time of day, atmospheric conditions, and scattering – short wavelengths more readily scattered – blue sky.
- Makes sense humans and other animals evolved optics tuned for light where the sun’s spectrum peaks.
- Short ultraviolet mostly blocked by the atmosphere, particularly the ozone layer – ionization effects and health hazard.

Example: Cosmic Microwave Background Radiation



Example: Cosmic Microwave Background Radiation

- EM radiation from the Big Bang – redshifted by the expansion of the universe, nearly isotropic.
- Corresponds to a blackbody radiation spectrum with a temperature of 2.73 K.
- Peak frequency = $2.82 k_B T / h = 2.82\ * \left(1.381 * 10^{-23}\ \frac{J}{K}\right)\ * (2.73\ \mathrm{K}) / \left(6.626 * 10^{-34}\ \mathrm{J{*}s}\right) = 1.60\ * 10^{11}\ Hz.$
- Peak Energy for Photons = $2.82 k_B T = 2.82 * \left(8.617 * 10^{-5}\ \frac{eV}{K}\right) * (2.73\ K) = 6.63\ * 10^{-4}\ eV.$

Example: Cosmic Microwave Background Radiation

- Peak Wavelength = $b/T = \ 2.898 \ * \ 10^{-3} \, mK/(2.73 \, K) = 1.06 * 10^{-3} \, m = 1.06 \, mm$ (283 giga Hz, 1.17 milli eV for photons) – short microwaves.

Example: Cosmic Microwave Background Radiation

- Energy density: Recall intensity = c/4 * (energy density), so
- Energy density $= \frac{4}{c} * I = \frac{4}{c} * 2\pi^5 k_B^4 T^4 / \left(15 h^3 c^2\right)$
- $= 8\pi^5 k_B^4 T^4 / \left(15 h^3 c^3\right) = 4\sigma T^4 / c$, where $\sigma \approx 5.670 * 10^{-8} \frac{W}{m^2 K^4}$
- Energy density $= \left(4 * \left(5.670 * 10^{-8} \frac{W}{m^2 K^4}\right) * (2.73\, K)^4\right) / \left(3.00 * 10^8 \frac{m}{s}\right)$
- $= 4.20 * 10^{-14} \frac{J}{m^3} = 0.262 \frac{eV}{cm^3}$

Quantum Statistics

- Blackbody spectrum can be derived from the photon gas model.
- Requires distributions which take quantum mechanical effects into account.
- $Z_{total} \approx \prod_{i=1}^{N} Z_i / N!$ is not valid because we multiple particles can occupy the same quantum state, and occupancy can affect the probabilities of other particles occupying the same state.
- Return to our set up of a small system in equilibrium with a much larger reservoir, but now we will allow particles as well as energy to be exchanged. The state is defined by both energy and the number of particles.
- System is vaguely defined – could be a particular quantum state.

Gibbs Factor

- Prob(s_1)/Prob(s_0) = Ω_{R_1}/ Ω_{R_0}= exp[(S_{R_1}- S_{R_0})/ k_B]
- $\approx \exp\left[\frac{1}{k_B}\left(\frac{\alpha S_R}{\alpha E_R} * -(\varepsilon_1 - \varepsilon_0) + \frac{\alpha S_R}{\alpha N_R} * -(n_1 - n_0)\right)\right] =$
- $= \exp\left[\frac{1}{k_B}\left(\frac{1}{T} * -(\varepsilon_1 - \varepsilon_0) + \frac{-\mu}{T} * -(n_1 - n_0)\right)\right]$
- $= \exp\left[\left(\mu(n_1 - n_0) - (\varepsilon_1 - \varepsilon_0)\right)/(k_B T)\right]$
- $= \exp\left[\beta\left(\mu(n_1 - n_0) \ - (\varepsilon_1 - \varepsilon_0)\right)\right]$
- $\mu \equiv -T \ * \frac{\alpha S}{\alpha N}$ is the chemical potential. Equal chemical potential means diffusive equilibrium.

Gibbs Factor

- Gibbs Factor = $\exp[\beta(\mu n_s - E_s)] \propto Prob(s)$
- $Prob(s) = \frac{1}{Ƶ} * \exp[\beta(\mu n_s - E_s)]$
- $Ƶ = \sum_s \exp[\beta(\mu n_s - E_s)]$, sum of the Gibbs factors, is the grand partition function.
- If more than one type of particle is present, then
- Gibbs Factor = $\exp\left[\beta\left(\left(\sum_j \mu_j * n_{j,s}\right) - E_s\right)\right]$.

Gibbs Factor

- Assume one type of particle, and the particles are indistinguishable.
- Let n be the number of particles in the quantum state, and let energy $\mathrm{E} = n\varepsilon$.
- Gibbs Factor $= \exp[\beta(\mu n_s \ - \varepsilon n_s)] = \exp[-n_s\beta(\varepsilon - u)]$
- $= (\exp[-\beta(\varepsilon - u)])^{n_s}$

Fermions

- Fermions (ex. Electrons) combine their wavefunctions anti-symmetrically – therefore the Pauli exclusion principle applies – no two fermions can have an identical set of quantum numbers.
- For the Gibbs factor, n can have only two values: 0 or 1.
- $\text{Ƶ} = (\exp[-\beta(\varepsilon - u)])^0 + (\exp[-\beta(\varepsilon - u)])^1$
- $= 1 + \exp[-\beta(\varepsilon - u)]$
- $Prob(n = 0) = \ 1/(1 + \exp[-\beta(\varepsilon - u)])$
- $Prob(n = 1) = \ \exp[-\beta(\varepsilon - u)]/(1 + \exp[-\beta(\varepsilon - u)])$
- $= \ 1/(\exp[\beta(\varepsilon - u)] + 1)$

Fermions

- $< n > = Prob(n = 1) = 1/(\exp[\beta(\varepsilon - u)] + 1)$ - Fermi-Dirac distribution.
- $< E > = \ \varepsilon \ * < n > = \varepsilon/(\exp[\beta(\varepsilon - u)] + 1)$
- For $\varepsilon \ll \ \mu, < n > \to 1$
- For $\varepsilon \approx \ \mu, < n > \approx \ \frac{1}{2}$
- For $\varepsilon \gg \ \mu, < n > \to 0$

Bosons

- Bosons (ex. Photons and other force particles) combine their wavefunctions symmetrically – therefore the Pauli exclusion principle is not applicable – any non-negative integer number of bosons can have an identical set of quantum numbers – tend to cluster.
- $n = \{0, 1, 2\ , \dots , \infty\}$
- $\text{Ƶ} = \sum_{j=0}^{\infty}(\exp[-\beta(\varepsilon - u)])^{j} = 1/(1 - \exp[-\beta(\varepsilon - u)])$,
- by the geometric sum, $\mu < \varepsilon$
- $< n > = \left(\sum_{j=0}^{\infty} j \ * \ \exp[-j\beta(\varepsilon - \mu)]\right) * (1 - \exp[-\beta(\varepsilon - u)])$

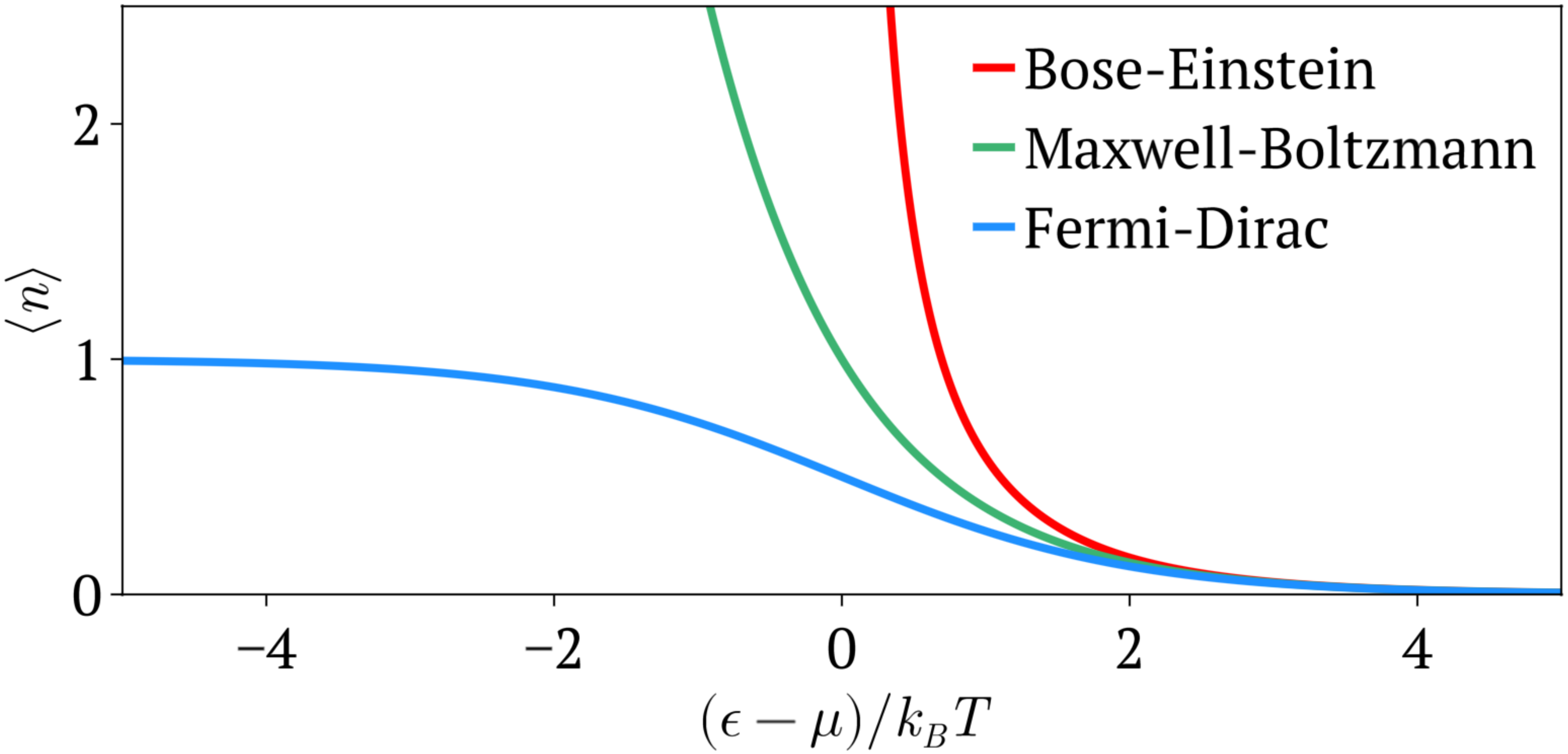
Bosons

- Let $\gamma = \beta(\varepsilon - \ \mu)$
- $< n > = \left(\sum_{j=0}^{\infty} j \ * \ \exp[-j\gamma]\right) * (1 - \exp[-\gamma])$
- $= \left(\sum_{j=0}^{\infty} -d(\exp[-j\gamma])/d\gamma\right) * (1 - \exp[-\gamma])$
- $= \left(-d\left(\sum_{j=0}^{\infty} \exp[-j\gamma]\right)/d\gamma\right) * (1 - \exp[-\gamma])$
- $= \left(-d\left(\sum_{j=0}^{\infty} \exp[-\gamma]^{j}\right)/d\gamma\right) * (1 - \exp[-\gamma])$
- $= (-d(1/(1 - \exp[-\gamma]))/d\gamma) * (1 - \exp[-\gamma])$

Bosons

- $< n > = \left(\exp[-\gamma]/(1 - \exp[-\gamma])^2\right) * (1 - \exp[-\gamma])$
- $= \ \exp[-\gamma]/(1 - \exp[-\gamma]) = \ 1/(\exp[\gamma] - \ 1)$
- $< n > = 1/(\exp[\beta(\varepsilon - \ \mu)\ ] - \ 1)$ - Bose-Einstein Distribution
- $< E > = \ \varepsilon * < n > = \varepsilon/(\exp[\beta(\varepsilon - \ \mu)\ ] - \ 1)$
- $\varepsilon > \ \mu$ required
- For $\varepsilon \ \rightarrow_{+} \ \mu, < n > \rightarrow \ \infty$
- For $\varepsilon \gg \ \mu, < n > \rightarrow 0$

Distributions Comparisons



Chemical Potential and Photons

- Chemical potential typically depends on temperature. Complicates the mathematics of quantum statistics.
- The chemical potential conserves the total number of particles. The total number of matter particles is not affected by temperature (aside from ultra high energy situations).
- There is no conservation law of photons. Their numbers change freely according to energy.
- Photons are therefore always in diffusive equilibrium, so their chemical potential is zero.

Distribution for Photons

- Photons are bosons, so they follow the Bose-Einstein distribution.
- Photon energy = Planck’s constant * frequency.
- Bose-Einstein distribution for photons, with chemical potential set to zero:
- $< n_{photons} > = 1/(\exp[\beta hf] - 1)$
- $< E_{photons} > = (hf)/(\exp[\beta hf] - 1)$

Blackbody Radiation Spectrum – Photon Gas

- Recall: $E = \pi \int_0^\infty < \varepsilon_n > n^2 dn.$ n here is for modalities, not number of particles.
- $< \varepsilon_n > =< E_{photons,\, n} > = (hf_n)/(\exp[\beta hf_n] - 1),$
- Where $f_n = (nc)/(2L).$
- $E = \pi \int_0^\infty ((nhc)/(2L))/(\exp[(nhc)/(2Lk_B T)] - 1) * n^2 dn$
- n = 2Lf/c = 2L/λ = 2Lε/(hc),
- dn = 2L/c * df = $-2L * d\lambda/\lambda^2$= 2L/(hc) * dε,
- where λ is the wavelength and ε = hf is the photon energy.

Blackbody Radiation Spectrum – Photon Gas

- $E = \pi \int_0^\infty (hf)/(\exp[(hf)/(k_B T)] - 1) * \left(\frac{2Lf}{c}\right)^2 * \frac{2L}{c} * df$
- $= L^3 \int_0^\infty 8\pi h/c^3 * f^3/(\exp[(hf)/(k_B T)] - 1) * df$
- $E = \pi \int_\infty^0 \left(\frac{hc}{\lambda}\right)/(\exp[(hc)/(k_B T\lambda)] - 1) * \left(\frac{2L}{\lambda}\right)^2 * \frac{-2L}{\lambda^2} * d\lambda$
- $= L^3 \int_0^\infty 8\pi hc * 1/\lambda^5 * 1/(\exp[(hc)/(k_B T\lambda)] - 1) * d\lambda$
- $E = \pi \int_0^\infty \varepsilon/(\exp[\varepsilon/(k_B T)] - 1) * \left(\frac{2L\varepsilon}{hc}\right)^2 * \frac{2L}{hc} * d\varepsilon$
- $= L^3 \int_0^\infty 8\pi/(hc)^3 * \varepsilon^3/(\exp[\varepsilon/(k_B T)] - 1) * d\varepsilon$

Blackbody Radiation Spectrum – Photon Gas

- Multiply energy by c/(4 * volume), where volume $= L^3$, for the intensity (power per area):
- $I = \int_0^\infty 2\pi h/c^2 * \ f^3/(\exp[(hf)/(k_B T)] - \ 1) \ * \ df$
- $I = \ \int_0^\infty 2\pi h * \ 1/\lambda^5 \ * 1/(\exp[(hc)/(k_B T\lambda)] - \ 1) * \ d\lambda$
- $I = \int_0^\infty 2\pi/\left(h^3 c^2\right) * \ \varepsilon^3/(\exp[\varepsilon/(k_B T)] - \ 1) \ * \ d\varepsilon$

Blackbody Radiation Spectrum – Photon Gas

- To integrate for the total intensity:
- $I = 2\pi(k_B T)^4/\left(h^3 c^2\right) \ * \ \int_0^\infty \gamma^3/(\exp[\gamma] - 1) * \ d\gamma$,
- where $\gamma \ = \frac{\varepsilon}{k_B T}, d\gamma = \ \frac{d\varepsilon}{k_B T}, \varepsilon = k_B T\gamma, d\varepsilon = k_B T \ * d\gamma$
- $\int_0^\infty \gamma^3/(\exp[\gamma] - 1) * d\gamma = \ \frac{\pi^4}{15}.$
- $I = 2\pi^5 k_B^4/\left(15h^3 c^2\right) \ * \ T^4$
- Energy density = $\frac{4}{c} * I = 8\pi^5 k_B^4 T^4/\left(15h^3 c^3\right)$

Blackbody Radiation – Number of Photons

- May be interested in the density of photons.
- $n_{photons} = \pi \int_0^\infty < n_{photons,n} > n^2 dn.$
- $< n_{photons,n} > \; = 1/(\exp[\beta h f_n] - \; 1),$
- Where $f_n = \; (nc)/(2L).$
- $< n_{photons,n} > = \pi \int_0^\infty 1/(\exp[(nhc)/(2Lk_B T)] - \; 1) * n^2 dn$
- n = 2Lf/c = 2L/λ = 2Lε/(hc),
- dn = 2L/c * df = $-2L \; * \; d\lambda/\lambda^2$= 2L/(hc) * dε,
- where λ is the wavelength and ε = hf is the photon energy.

Blackbody Radiation – Number of Photons

- $n_{photons} = \pi \int_0^\infty 1/(\exp[(hf)/(k_B T)] - 1) * \left(\frac{2Lf}{c}\right)^2 * \frac{2L}{c} * df$
- $= L^3 \int_0^\infty 8\pi/c^3 * f^2/(\exp[(hf)/(k_B T)] - 1) * df$
- $n_{photons} = \pi \int_\infty^0 1/(\exp[(hc)/(k_B T\lambda)] - 1) * \left(\frac{2L}{\lambda}\right)^2 * \frac{-2L}{\lambda^2} * d\lambda$
- $= L^3 \int_0^\infty 8\pi * 1/\lambda^4 * 1/(\exp[(hc)/(k_B T\lambda)] - 1) * d\lambda$
- $n_{photons} = \pi \int_0^\infty 1/(\exp[\varepsilon/(k_B T)] - 1) * \left(\frac{2L\varepsilon}{hc}\right)^2 * \frac{2L}{hc} * d\varepsilon$
- $= L^3 \int_0^\infty 8\pi/(hc)^3 * \varepsilon^2/(\exp[\varepsilon/(k_B T)] - 1) * d\varepsilon$

Blackbody Radiation – Number of Photons

- Divide by volume, where volume $= L^3$, for the density:
- $[n_{photons}] = \int_0^\infty 8\pi/c^3 * \; f^2/(\exp[(hf)/(k_B T)] - \; 1) \; * \; df$
- $[n_{photons}] = \; \int_0^\infty 8\pi * \; 1/\lambda^4 \; * 1/(\exp[(hc)/(k_B T\lambda)] - \; 1) * \; d\lambda$
- $[n_{photons}] = \int_0^\infty 8\pi/(hc)^3 * \; \varepsilon^2/(\exp[\varepsilon/(k_B T)] - \; 1) \; * \; d\varepsilon$

Blackbody Radiation – Number of Photons

- To integrate for the total density:
- $\left[n_{photons}\right] = 8\pi((k_B T)/(hc))^3 * \int_0^\infty \gamma^2/(\exp[\gamma] - 1) * d\gamma$,
- where $\gamma = \frac{\varepsilon}{k_B T}, d\gamma = \frac{d\varepsilon}{k_B T}, \varepsilon = k_B T\gamma, d\varepsilon = k_B T * d\gamma$
- $\int_0^\infty \gamma^2/(\exp[\gamma] - 1) * d\gamma \approx 2.40$, from numeric integration, no known analytic method.
- $\left[n_{photons}\right] \approx 8\pi(k_B/(hc))^3 * 2.40 * T^3$
- $= 19.20 * \pi(k_B/(hc))^3 * T^3$

Example: Cosmic Microwave Background Radiation

- Temperature = 2.73 K.
- Photon Density = $19.20 \; * \; \pi(k_B/(hc))^3 * T^3$
- $= 19.20 \; * \; \pi\left(\left(8.617 * 10^{-5} \; \frac{eV}{K}\right) / \left(\left(4.136 \; * \; 10^{-15} \, eV \; * s\right) * \; \left(3.00 \; * 10^8 \; \frac{m}{s}\right)\right)\right)^3 \; * \; (2.73 \; K)^3 \; =$ 411 million photons per m^3 = 411 photons per cm^3.

Example: Cosmic Microwave Background Radiation

- Average photon energy = (energy density)/(photon density).
- Average photon energy ≈ $\left(8\pi^5 k_B^4 T^4/\left(15h^3c^3\right)\right)/\left(2.40 * 8\pi(k_B T/(hc))^3\right)$
- $= \ \pi^4/36.0 \ * \ k_B T \approx 2.71 * \ k_B T$
- For T = 2.73 K, average photon energy $\approx 2.71 * \ k_B T$
- $= 2.71 \ * \left(8.617 * 10^{-5} \ \frac{eV}{K}\right) \ * \ (2.73\ K) = 6.37 \ * 10^{-4}\ eV$
- Note: similar to, but not identical to, the peak photon energy:
- $2.82 * \ k_B T \rightarrow \ 6.63 \ * 10^{-4}\ eV$ for T = 2.73 K.

Quantum Revolution

- Max Plank's quantization of energy resolved the ultraviolet catastrophe and made it possible to derive the properties of blackbody radiation from first principles.
- Beginning of quantum mechanics.
- Energy levels of electrons in atoms orbiting the atomic nucleus also quantized.
- Explains spectral lines: only photons with energy corresponding to discrete jumps are absorbed or omitted.

Quantum Revolution

- Explained the photo-electric effect – one of Albert Einstein’s 1905 papers - won him the 1921Nobel Prize in Physics.
- Light only ejects electrons from metal if the frequency is high enough. No electrons are ejected if the frequency is too low, regardless of intensity.
- Individual photons must carry enough energy to eject electrons.

Sources

- Dr. James A. McNeil – Professor of Physics at Colorado School of Mines – taught the thermal physics course I took as an undergraduate physics student there.
- Daniel V. Schroeder. An Introduction To Thermal Physics. – textbook for my thermal physics course.
- “What is the Ultraviolet Catastrophe”. Physics Explained. YouTube channel. - PhD theoretical physicist according to the channel description – inspiration for the lecture.
- Note: For “What is the Ultraviolet Catastrophe”. Physics Explained., I take issue with how the author presents deriving the classical average energy for an EM wave modality. Otherwise mostly excellent.

Sources

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- Wikipedia.
- Jacob Bronowski. The Ascent of Man.