Basis vectors and their transformations

Here are some things to be aware of regarding basis vectors and their transformations. This page has the background and then I'll do the specific example of polar coordinates on the next page.

Vectors project onto one-forms, one-forms onto vectors

In other words, we project a vector onto its basis like this:

$$[v^\alpha] = v^\alpha e_\alpha$$

And a one-form like this:

$$[v_\alpha] = v_\alpha e^\alpha$$

In the above expressions, v^α and v_α represent the α^{th} component of the vector, as usual. However e^α and e_α are not components. They represent the entire α^{th} basis vector.

The syntax for raising and lowering basis vectors

Here we take the basis for vectors and raise it into a basis for one-forms:

$$e^\alpha = g^{\alpha\beta} e_\beta$$

Again, e^α and e_β are not components, they are entire vectors. So, in two dimensions, this expands to:

$$e^1 = g^{11} e_1 + g^{12} e_2$$

$$e^2 = g^{21} e_1 + g^{22} e_2$$

Tensor transformations respect the structure of the metric tensor

We haven't gotten to this yet, but what I'm refering to is that the metric tensor consists of dot products of the basis vectors:

$$g_{\alpha\beta} = e_\alpha \bullet e_\beta$$

All this means that our basis vectors are not, in general, normalized. I believe these (non-normalized) bases are sometimes called "coordinate bases" as opposed to "orthonormal bases."

Polar Coordinates

Click here for a cheat-sheet with all the x,y to polar transformations and so on.

(And please report any errors!)

The basis for vectors consists of one-forms. In Cartesian coordinates there's no difference between the two. But when we do the translation to polar we need to take that into account.

$$e_r = \frac{\partial x}{\partial r} e_x + \frac{\partial y}{\partial r} e_y = \cos\theta\, e_x + sin\theta\, e_y = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \hat{r}$$

$$e_\theta = \frac{\partial x}{\partial \theta} e_x + \frac{\partial y}{\partial \theta} e_y = -r\sin\theta\, e_x + r\cos\theta\, e_y = r \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = r\hat{\theta}$$

So e_r and e_θ are orthogonal but e_θ is not unit length.

These tensor transformations produce basis vectors that are consistent with the metric definition, in particular:

$$g_{\theta\theta} = e_\theta \bullet e_\theta = r^2(\sin^2\theta + \cos^2\theta) = r^2$$

To have a basis for one-forms we have to raise the indices on e_r and e_θ:

$$g_{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}, \quad g^{r\theta} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$$

$$e^r = g^{rr} e_r + g^{r\theta} e_\theta = (1)\,\hat{r} + (0)\,e_\theta = \hat{r}$$

$$e^\theta = g^{\theta r} e_r + g^{\theta\theta} e_\theta = (0)\,\hat{r} + \frac{1}{r^2}\, r\hat{\theta} = \frac{1}{r}\,\hat{\theta}$$

And this explains why we don't have a $\frac{1}{r}$ in the gradient definition, because in these non-normalized polar coordinates it's part of the θ basis vector.