1. Tensors have rank, dimension, and transform properly

- **Rank**: The shape of a tensor, noted by the number of indices
  - Scalars are rank 0
  - Vectors and one-forms are rank 1
  - Matrices are rank 2
- **Dimension**: The number of components within each index
  - This is your classic understanding of "dimension", 2-D, 3-D, etc.
  - 4-dimensional Minkowski space: (t, x, y, z)
  - 3-dimensional spherical space: (r, θ, ϕ)
- **Transformation**:
  - Underlying physical and geometric qualities invariant under coordinate transformation
  - The representation may change without impacting the real structure

2. Rank 1 tensors

- **Vectors**: Upper-indexed rank 1 tensors, $\left[x^{\mu}\right]$
  - Conceptualize as vectors on a tangent hyperplane to a surface or infinitesimal displacement vectors on a surface
  - 3-D position vector: $\left[x^{0}, x^{1}, x^{2}\right]$
  - 2-D velocity vector: $\left[\frac{dx^{0}}{dt}, \frac{dx^{1}}{dt}\right]$
- **One-forms**: Lower-indexed rank 1 tensors, $\left[x_{\mu}\right]$, $\left[\Phi_{\mu}\right]$, etc
  - These capture how scalar values change over a surface, like gradients
  - For a scalar function Φ, one-forms could be defined as:
    - General 3-D one-form: $\left[\frac{\partial\Phi}{\partial x^{0}}, \frac{\partial\Phi}{\partial x^{1}}, \frac{\partial\Phi}{\partial x^{2}}\right]$
    - Radial 2-D one-form: $\left[\frac{\partial\Phi}{\partial r}, \frac{\partial\Phi}{\partial \theta}\right]$
  - This is your classic understanding of "dimension", 2-D, 3-D, etc.
  - 4-dimensional Minkowski space: (t, x, y, z)
  - 3-dimensional spherical space: (r, θ, ϕ)

3. Metric tensor

- Rank 2 tensor that represents intrinsic curvature in space
- Can calculate as $g_{ij} = \sum_k \frac{\partial \bar{x}^k}{\partial x^i} \frac{\partial \bar{x}^k}{\partial x^j}$
- Metric tensors are invertible, $g_{\mu\nu} g^{\mu\nu} = I$, and $g^{ij} g_{jk} = \delta^i_k$
- 3-D spherical metric tensor:

$g_{\mu\nu} =$

1	0	0
0	r^2	0
0	0	$r^2 sin^2\theta$

$g^{\mu\nu} =$

1	0	0
0	$\frac{1}{r^2}$	0
0	0	$\frac{1}{r^2 sin^2\theta}$

4. Raising and lowering indices

- Transform between one-forms and vectors using the metric tensor
- $x_\mu = g_{\mu\nu} x^\nu$
- Radial coordinate example:

$$x_r = g_{rr} x^r + g_{r\theta} x^\theta + g_{r\phi} x^\phi = (1)x^r + (0)x^\theta + (0)x^\phi = x^r$$

$$x_\theta = g_{\theta r} x^r + g_{\theta\theta} x^\theta + g_{\theta\phi} x^\phi = (0)x^r + \left(r^2\right)x^\theta + (0)x^\phi = r^2 x^\theta$$

$$x_\phi = g_{\phi r} x^r + g_{\phi\theta} x^\theta + g_{\phi\phi} x^\phi = (0)x^r + (0)x^\theta + \left(r^2 sin^2\theta\right)x^\phi = r^2 sin^2\theta x^\phi$$

5. Transforming between coordinate systems

- Vectors
  - $\bar{x}^{\alpha} = \frac{\partial \bar{x}^{\alpha}}{\partial x^{\beta}} x^{\beta}$
  - Specifically changing polar to cartesian:
    $x = \frac{\partial x}{\partial r} r + \frac{\partial x}{\partial \theta} \theta$
    $y = \frac{\partial y}{\partial r} r + \frac{\partial y}{\partial \theta} \theta$
- One-forms
  - $\bar{\Phi}_{\alpha} = \frac{\partial x^{\beta}}{\partial \bar{x}^{\alpha}} \Phi_{\beta}$, where $\Phi_{\beta} = \frac{\partial \Phi}{\partial x^{\beta}}$ and Φ is a scalar function
  - In 2-D, generally
    $\bar{\Phi}_{\alpha} = \frac{\partial x^{0}}{\partial \bar{x}^{\alpha}} \Phi_{0} + \frac{\partial x^{2}}{\partial \bar{x}^{\alpha}} \Phi_{1}$
    $\bar{\Phi}_{\alpha} = \frac{\partial x^{0}}{\partial \bar{x}^{\alpha}} \frac{\partial \Phi}{\partial x^{0}} + \frac{\partial x^{1}}{\partial \bar{x}^{\alpha}} \frac{\partial \Phi}{\partial x^{1}}$
  - Specifically changing polar to cartesian:
    $\bar{\Phi}_{x} = \frac{\partial r}{\partial x} \frac{\partial \Phi}{\partial r} + \frac{\partial \theta}{\partial x} \frac{\partial \Phi}{\partial \theta}$
    $\bar{\Phi}_{y} = \frac{\partial r}{\partial y} \frac{\partial \Phi}{\partial r} + \frac{\partial \theta}{\partial y} \frac{\partial \Phi}{\partial \theta}$

Example

Raising and lowering indices in spherical coordinates (Collier 145, Problem 5.2)

a. Transform the vector $[A^{a}] = (1, r, 0)$ into a one-form $[A_{a}]$

$A_{a} = g_{ab} A^{b}$

$A_{r} = g_{rr} A^{r} + g_{r\theta} A^{\theta} + g_{r\phi} A^{\phi} = g_{rr} A^{r} = (1)(1) = 1$

$A_{\theta} = g_{\theta r} A^{r} + g_{\theta\theta} A^{\theta} + g_{\theta\phi} A^{\phi} = g_{\theta\theta} A^{\theta} = (r^{2})(r) = r^{3}$

$A_{\phi} = g_{\phi r} A^{r} + g_{\phi\theta} A^{\theta} + g_{\phi\phi} A^{\phi} = g_{\phi\phi} A = (r^{2} sin^{2}\theta)(0) = 0$

$[A_{a}] = (1, r^{3}, 0)$

b. Transform the one-form $[B_{a}] = (0, - r^{2}, cos^{2}\theta)$ into a vector $[B^{a}]$

$B^{a} = g^{ab} B_{b}$

$B^{r} = g^{rr} B_{r} + g^{r\theta} B_{\theta} + g^{r\phi} B_{\phi} = g^{rr} B_{r} = (1)(0) = 0$

$B^{\theta} = g^{\theta r} B_{r} + g^{\theta\theta} B_{\theta} + g^{\theta\phi} B_{\phi} = g^{\theta\theta} B_{\theta} = \left(\frac{1}{r^{2}}\right)(- r^{2}) =$ -1

$B^{\phi} = g^{\phi r} B_{r} + g^{\phi\theta} B_{\theta} + g^{\phi\phi} B_{\phi} = g^{\phi\phi} B_{\phi} = \left(\frac{1}{r^{2} sin^{2}\theta}\right)(cos^{2}\theta) = \frac{cot^{2}\theta}{r^{2}}$

$[B^{a}] = \left(0, - 1, \frac{cot^{2}\theta}{r^{2}}\right)$