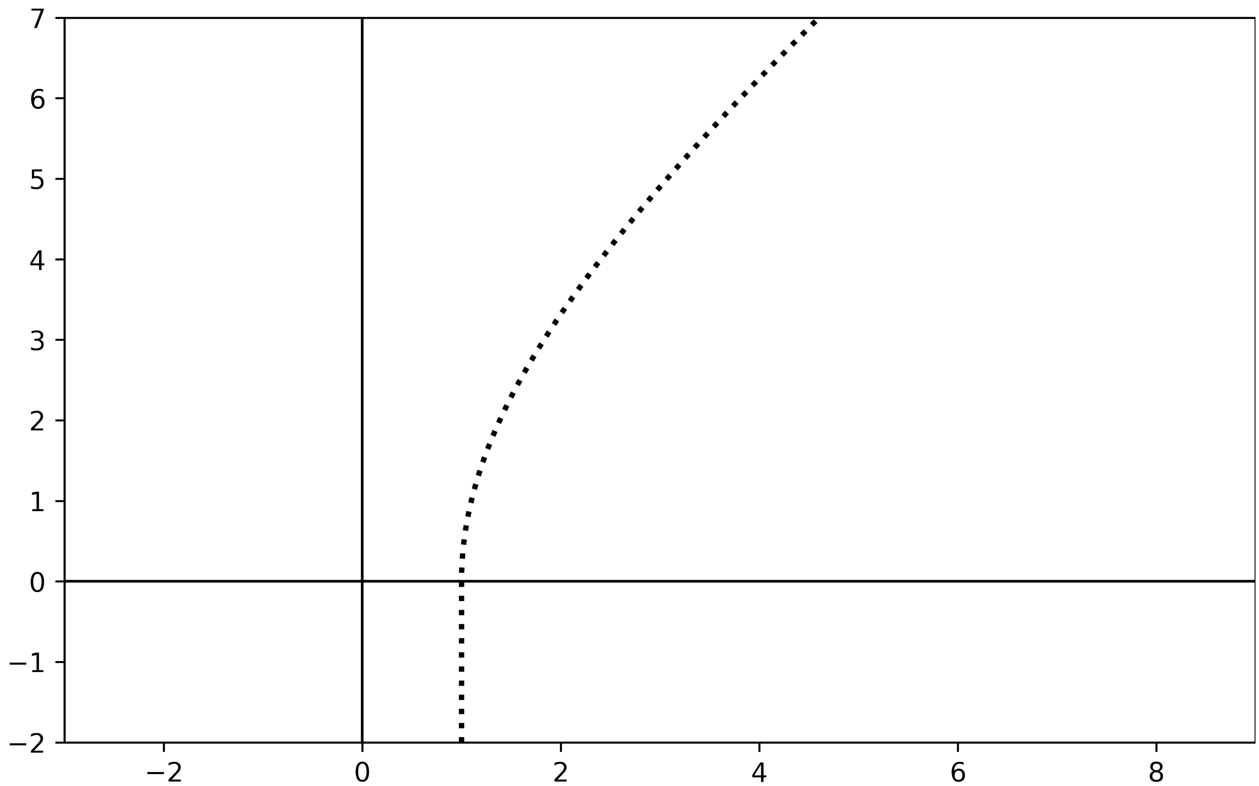
Acceleration

General Plan

1. Spacetime diagram of an accelerating object.
2. Momentarily comoving reference frames.
3. The relationship of time between the accelerated observer and other frames.
4. Two accelerating observers.
5. Born rigid motion (straight line).
6. GR: The Equivalence Principle
7. Can we extend the analogy between rotation in space and spacetime for acceleration?
   If so, how does this compare to Born motion?
   It seems to me that gravitational time dilation would NOT be an additional part of the calculation.
8. Dewan and Beran's 1959 paper.
9. John Bell's 1976 paper.
10. Acceleration vs. spacetime paths (Maudlin). Diagrams of that.
11. The twins scenario.
12. (More exercises?)
13. References.

(1) An accelerating object



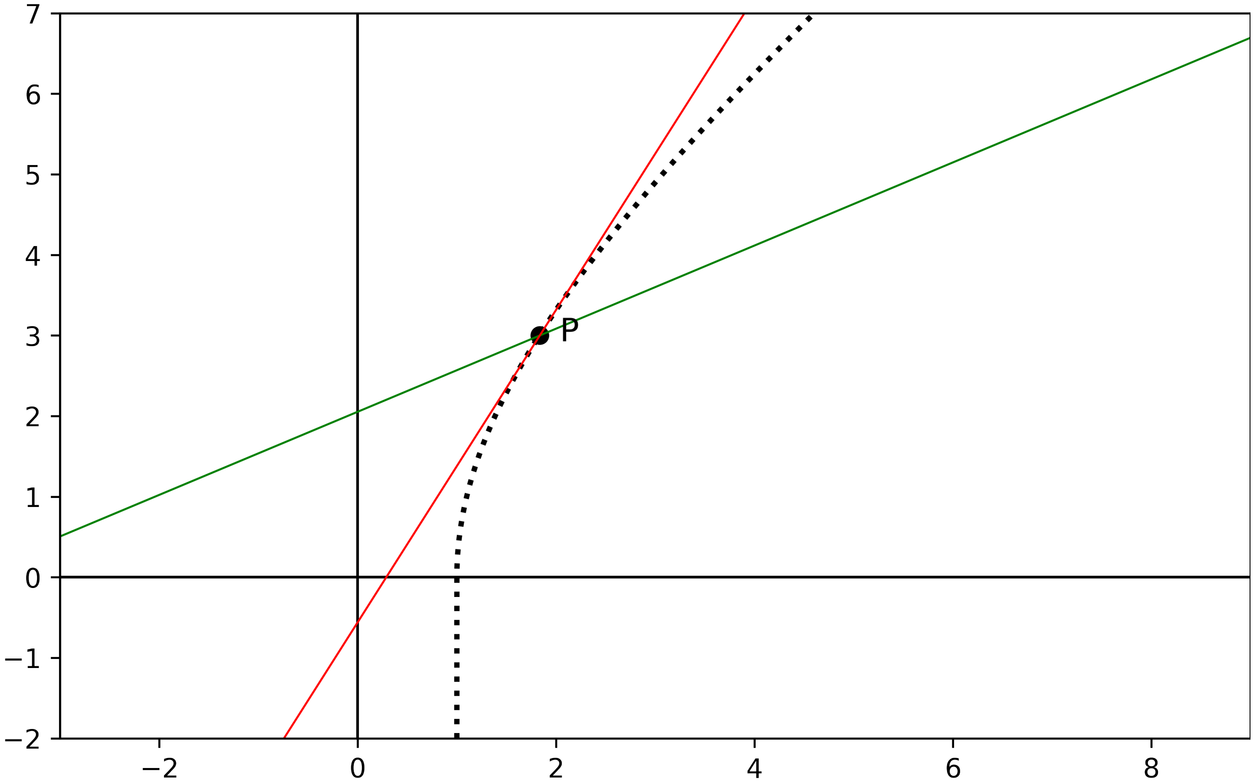
Here we see the path of an object that begins accelerating at time zero, from the perspective of an inertial frame from which we are observing it. The object has an inertial (straight line) path up until $t = 0$ and then it begins to bend, as the object accelerates.

What we see here is constant acceleration. Nevertheless, the path straightens out after a while, because no matter how much the object accelerates it will never reach the speed of light. So the path will eventually come close to 45 degrees, but never quite get there.

The term *proper acceleration* refers to that acceleration "felt" by the observer who is accelerating. In other words, proper acceleration is what that observer's accelerometer would register. When I say the path above has constant acceleration, I mean that the proper acceleration is constant. In our observing frame the acceleration will diminish over time, as it must if we are not to see the object surpass the speed of light.

Proper acceleration is typically called either α or a_0. In what follows I'll use a_0.

(2) Momentarily comoving reference frames



We can pick a point (P) on the accelerating path, and then calculate the velocity of the accelerating object (with respect to the inertial frame from which we are viewing it) at that instant.

Then we can talk about an inertial frame who's velocity matches that of the accelerating object at that instant. This matching inertial frame is called the *momentarily comoving reference frame* or the MCRF.

We can use the MCRF to make various calculation about things from the perspective of the accelerating observer at that particular instant. For example, here I've drawn the time axis (red) and the x axis (green).

(3) The accelerated observer and other reference frames

Here are the basic equations:

$$(1)\ \ \Delta x(t) = \left(\sqrt{1 + a_0^2 t^2} - 1\right) \Big/ a_0$$

$$(2)\ \ a(t) = a_0 \Big/ \left(1 + a_0^2 t^2\right)^{3/2}$$

$$(3)\ \ v(t) = a_0\, t \Big/ \left(1 + a_0^2 t^2\right)^{1/2}$$

$$(4)\ \ t_0 = \frac{1}{a_0} \sinh^{-1}(a_0\, t)$$

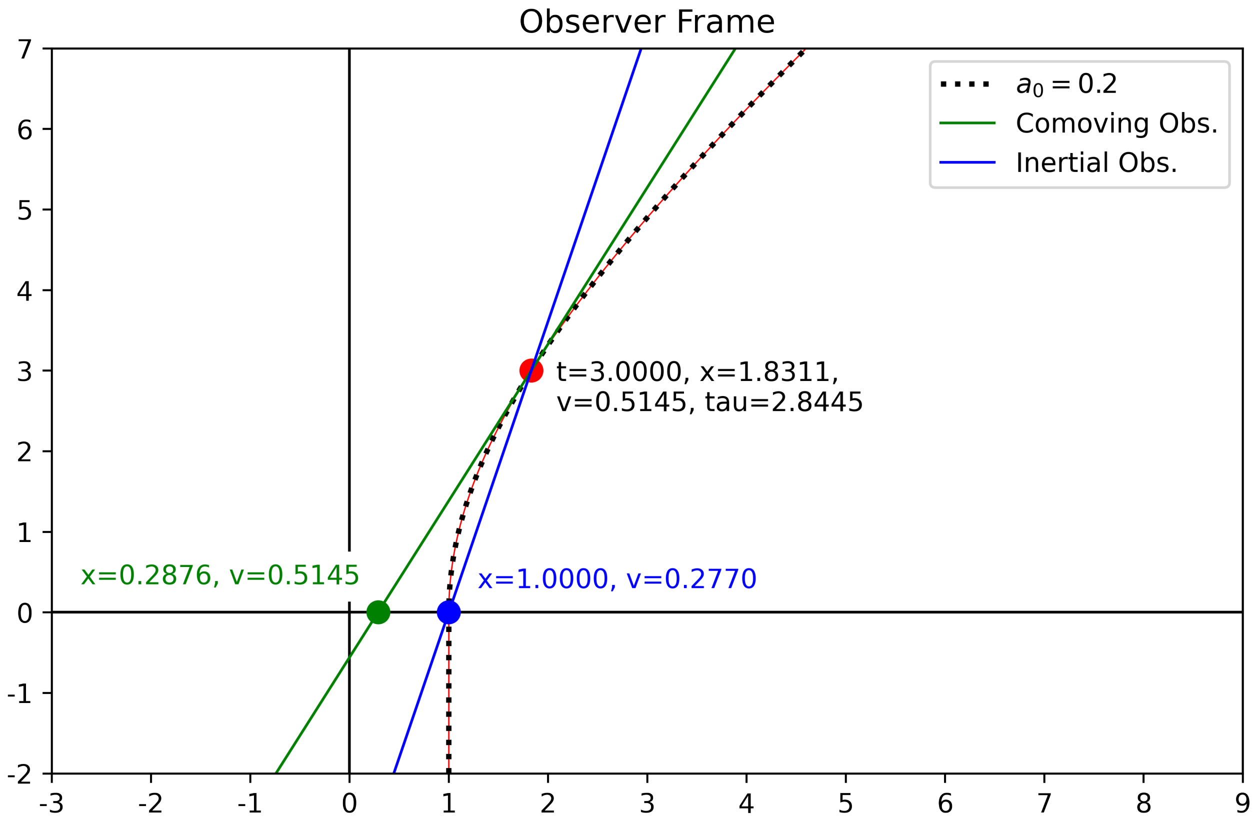
$$(5)\ \ t = \frac{1}{a_0} \sinh(a_0\, t_0)$$

- a_0 is the proper acceleration of the object.
- t is the time in the inertial frame from which we're looking at the path.
- The object starts accelerating at $t = 0$ (from any x position).
- $\Delta x(t)$ is how far the object has moved from its starting point at time t.
- $v(t)$ is the velocity of the object as seen from the observing frame at time t.
- t_0 is the proper time of the accelerating object.

Equation (1) can be used to draw the path of the object. You need to do the calculation for each point on the plot to get the t and x coordinates for each dot. (2) and (3) will tell you what the acceleration and velocity of the object are, at each point, in the observing frame. We'll be using the velocity shortly. (4) and (5) can be used to convert back and forth between the time in the observing frame and the proper time for the accelerating object.

We previously defined the proper time between two points as the time of an inertial path between those points, and said that was the shortest time. But *accelerated observers* also have a proper time between any two points on their path, and this time will be *shorter* than that of an inertial path between those points.

The proper time on any timelike path, including an accelerated one, is equal to the total length of the path. The total length of an accelerated path is invariant. *I guess A light path is not a timelike path in this sense.*



Observer time at red dot = 3.0000

Comoving time at red dot = 2.5725

Inertial time at red dot = 2.8826

Accelerating time at red dot = 2.8445

In this picture, the object began accelerating at from $x = 1$ at $t = 0$. I've picked a point on the accelerated path and put a red dot there. Then drawn a time axis for the CMRF at that point. There's a green dot at the comoving frame's $t = 0$. Then I've drawn a blue line for an inertial frame which starts out at the same time and position as the accelerating frame (at the blue dot) and has the velocity needed to meet the accelerating frame at the red dot.

Exercise Set One

For all of these exercises, we are assuming an inertial "observing frame" from which we are looking at all the other (accelerated and inertial) paths. The diagram we are going to draw is the picture in this observing frame. For clarity, I'm going to give these paths colors. The accelerated path is the red one. The inertial path comoving with the accelerated path is the green one and the other inertial path is the blue one. You can use those colors to draw them if you like, or you can just label them.

When I ask for time, positions, and so on, they are meant to be in the observing frame unless otherwise specified.

I'm going to specify an acceleration of, for example .2. This is two tenths of the speed of light. But we're using the same units for time and space, so c=1 and we just say .2. Likewise, any time or space locations are just given as numbers. It doesn't actually matter what units you think of them as being in, as long as time and space have the same units.

1. Consider a rocket which starts from the observing frame at $t = 0$ and $x = 2$ with constant proper acceleration $a_0 = .2$. Use Equation (1) above to plot the rocket's positions at times 2, 4, and 6. Use those points to sketch the path of the rocket.

2. Now take the point on the path at $t = 4$. Put a red dot on the accelerating path there. Use Equation (3) to find the velocity of the rocket at that point.

3. We want to construct the green world line of the comoving inertial path at that same point. In order to do this, use the velocity you found to determine at which x location this path crosses $t = 0$ in the diagram. Put a green dot at that location and draw the green path so it passes through the green and red dots.

4. Now we want to construct a blue path that we can use to calculate the time seen by an inertial observer (that is, moving at constant velocity) who is present when the rocket starts accelerating and is also present at the red dot. Draw the path by putting a blue dot where the acceleration begins and drawing the blue path so it passes through the blue and red dots.

5. Given the time and distance covered between the blue and red dots, calculate the velocity of the blue observer.

6. Find the time of the red dot in the green observer's frame of reference. You can do this by calculating the spacetime interval between the green and red events. In the green frame, the observer would be moving up their timeline, so this would be their proper time. But since the interval is invariant, you can calculate it in the observing frame using the values we already have.

7. Find the time of the red dot in the blue observer's frame of reference. Again, you can use the spacetime interval. This will be the proper time seen by the blue observer between the blue and red events.

8. Use Equation (4) to find the proper time interval seen by the accelerating observer between the blue and red events. This should come out to be less than the blue observer's time.

NOTE:
https://www.wolframalpha.com can calculate sinh() and arcsinh()